Week 05 Monday 09/15 and Wednesday 09/17

Read 4.2, 4.3

Exercises 1.4: 7 | 4.1: 8, 22 | 4.2: 1, 2, 3, 4, 5

Monday This week, lecture was covered by Professor Emily Witt. We started by reviewing several of the notions and results discussed last week. In particular, we motivated the naming of a T-cyclic subspace. We also gave examples of T-invariant subspaces, which relates to the notion of eigenvectors and eigenvalues. We also gave a sketch of the argument that given a linear operator V on a finite dimensional vector space V of dimension n, then T has a minimal polynomial, and its degree is at most n^2. Next, toward relating minimal polynomials, we proved the following **Theorem**: Suppose that T is cyclic and $\langle T, \mathbf{v} \rangle = V$. Given $g(x) \in \mathsf{k}[x]$, let $\mathbf{w} = g(T)(\mathbf{v})$ and let $d(x)$ denote the greatest common divisor of $\mu_{V,\mathbf{v}}(x)$ and $g(x)$. Then $\mu_{T,\mathbf{w}}(x) = \mu_{T,\mathbf{v}}(x)/d(x)$. Next, we stated the following **Theorem**: Suppose that T is cyclic. Given any T-invariant subspace W of V, there exists $\mathbf{w} \in V$ such that $W = \langle T, \mathbf{w} \rangle$, and $\mu_{T,\mathbf{w}}(x) \mid \mu_T(x)$. Moreover, given a monic divisor $g(x)$ of $\mu_T(x)$, there is a unique and T-invariant subspaces W of V such that $\mu_{T|_W}(x) = g(x)$. We illustrated some conclusions of the theorem in an example where $V = \mathbb{R}^2$.

Wednesday We started class with a quiz. Next, we recalled the theorem comparing minimal polynomials from last time, and the theorem on T-invariant subspaces. We proved the latter theorem using the former, besides leaving the uniqueness in the final statement as an exercise. Next, we defined what it means for a vector space V to be the *direct sum* of subspaces $U_1, \ldots, U_k$; we write $V = U_1 \oplus \cdots U_k$ in this case. We went through some examples, and a non-example, of direct sums.

Week 04 Monday 09/08 and Wednesday 09/10

Read 4.1

Exercises 4.1: 1, 2, 3, 4

Monday This week, lecture was covered by Professor Dan Katz. To start, we had our first quiz, which covered basic properties of polynomial division (the quiz was similar to Problem 3.1.10 in our book). We considered the following general **Question**: Given a finite dimensional V, a linear transformation $\mathsf{T} : \mathsf{V} \to \mathsf{V}$, and a vector $\mathbf{v} \in \mathsf{V}$, can we build an operator from T in a natural way that maps $\mathbf{v}$ to $\mathbf{0}$? What can we say about all such operators built from T? We addressed this by establishing the following **Theorem**: There exists a polynomial $0 \neq p(x) \in \mathsf{k}[x]$ such that $p(\mathsf{T})(\mathbf{v}) = \mathbf{0}$. Furthermore, there exists a *least such polynomial*, that is, a monic polynomial $\mu_{\mathsf{T},\mathbf{v}}(x) \in \mathsf{k}[x]$ with the property that $\mu_{\mathsf{T},\mathbf{v}}(\mathsf{T})(\mathbf{v}) = \mathbf{0}$ and such that $\mu_{\mathsf{T},\mathbf{v}}$ divides any other polynomial $p(x) \in \mathsf{k}[x]$ with $p(\mathsf{T})(\mathbf{v}) = \mathbf{0}$. This is called the *minimal polynomial of* T *with respect to* $\mathbf{v}$. We then carefully went over an example of how to compute this in a specific case.

We then considered what Cooperstein calls the T-*cyclic subspace of* V *spanned by* $\mathbf{v}$, which we may simply call the *cyclic subspace of* V *generated by* T *and* $\mathbf{v}$, or even more simply, we may omit the term *cyclic*. We established the basic properties of such subspaces (e.g., we described how to compute a basis in terms of powers of T applied to v).

After this, we introduced the notion of the *minimial polynomial of* T. Notice that the main difference is that this concept does not depend on any choice of $\mathbf{v} \in \mathsf{V}$. This is the least (monic) polynomial $\mu_{\mathsf{T}}(x) \in \mathsf{k}[x]$ with the property that $\mu_{\mathsf{T}}(\mathsf{T})$ is the zero operator $0_{\mathsf{V}} : \mathsf{V} \to \mathsf{V}$. In lecture, we discussed a proof of their existence.

Wednesday See the previous entry, which covered the contents of *both* of this week's lectures.

Week 03 Monday 09/01 and Wednesday 09/03

Read 3.2, 4.1

Exercises 3.2: 1, 2, 3, 4, 5

Monday No lecture. Happy Labor Day!

Wednesday We started by finishing up with a leftover from §3.1, the topic of *least common multiple* of a pair of polynomials. Given polynomials $f, g \in \mathsf{k}[x]$, a least common multiple is a polynomial $m \in \mathsf{k}[x]$ such that $f|m, g|m$, and such that m has the least possible degree. We also saw how this form of minimality is the same as saying that m divides any common multiple of f, g. After this, we recalled the definition of a root of a polynomial, and proved that $\lambda \in \mathsf{k}$ is a root of $f \in \mathsf{k}[x]$ if and only if $x - \lambda$ divides f in $\mathsf{k}[x]$. We introduced the concept of the *multiplicity* of a root, and proved that if $0 \neq f$, then f has at most $\deg(f)$ many roots, counting multiplicity. We defined what it means for a field to be *algebraically closed*, and then went over many non-examples. We also presented the *Fundamental Theorem of Algebra*, which states that $\mathbb{C}$ is algebraically closed. Note: It may be that this is the only algebraically closed field that students are familiar with, and throughout the semester, not much will be lost if "algebraically closed" is replaced with "the complex numbers". We saw that the only irreducible polynomials over an algebraically closed field are the linear polynomials. We then turned our attention to the real numbers. Using basic properties of *complex conjugation*, we proved that the if $f \in \mathbb{R}[x]$ is irreducible, then either f is linear, or f is quadratice, that is, has degree 2. We concluded by observing that a quadratic polynomial in $\mathbb{R}[x]$ is irreducible if and only if it has no root in $\mathbb{R}$, a condition that can be checked by applying the quadratic equation.

Week 02 Monday 08/25 and Wednesday 08/27

Read Read Cooperstein 3.1, 3.2

Exercises Cooperstein 3.1: 1 (assume the ambient field is $\mathbb{Q}$), 5, 8, 10

Monday We started lecture by observing that the vector space space $\mathcal{L}(V) = \mathcal{L}(V, V)$ of *endomorphisms* of V possesses additional structure. More precisely, we observed that operators in this vector space can be multiplied (composed). Motivated by this, we introduced the defnition of an *associated k-algebra*, and observed that $\mathcal{L}(V)$ is such an object, as is the vector space of square matrices, and also the vector space of polynomials $\mathsf{k}[x]$ in the variable x. The rest of lecture was dedicated to polynomials. After introducing notation, and examples, we stated, and proved, the *Division Algorithm* for polynomials over a field, and went over examples. We introduced what it means for a polynomial to divide another, and then recalled the definition of a greatest common divisor of two polynomials (not both zero). We then recalled the *Euclidean Algorithm* for computing a greatest common divisor, and went over an example. In that example, we saw that the greatest common divisor of polynomials $f, g \in \mathsf{k}[x]$ can be written as $a(x)f(x) + b(x)g(x)$ with $a(x), b(x) \in \mathsf{k}[x]$. This provided an example of the so-called *Bezout Identity*. We will discuss applications of this identity next lecture.

Wednesday We reviewed the basic facts about polynomials we proved last lecture, and then proved that the definition of the *greatest common divisor* of two polynomials, not both zero, agreed with that presented in Cooperstein. We then carefully went over an example of Bezout's Identity, and defined what it means for a polynomial to be *irreducible*, and what it means for two polynomials to be *relatively prime*. We applied Bezout's Identity to demonstrate that irreducible polynomials behave much like prime integers. This culminated in us stating, and mostly proving, the *Unique Factorization Theorem* for polynomials over a field.

Week 01 Monday 08/18 and Wednesday 08/20

Read Review Cooperstein 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 2.1, 2.2, 2.4, 2.5, 2.6

Exercises None this week, focus on skimming the book

Monday Welcome to Math 790! The first portion of lecture was dedicated to the syllabus and introductions. After this, we began our review of the basic facts of linear algebra, i.e., material from MATH 590. We recalled the definition of an abstract vector space, and presented examples. We also recalled the concepts of linear independence and span. We then stated the Exchange Lemma, and recalled the definition of, and basic facts about, the dimension of vector spaces. We talked a bit about infinite dimensional vector spaces at the end, but really, most of the time this semester, we will focus on the finite dimensional case.

Special assignment: Read our syllabus, and introduce yourself to me via email. Do this ASAP, and please address the following.

- If you are an undergraduate, are you attempting to use MATH 790 and 791 as a sequence for majors?
- If you are a non-math graduate student, is this course necessary for your program? What is your program?
- If you are a math graduate student, do you intend to take the qualifying exam in algebra?
- Your math background. In particular, have you satisfied the prerequisites for this course?
- If you were previously my student, a brief update on what you've been up to since then.
- Any future aspirations involving math. For instance, do you plan to attend graduate school, or pursue a career, in a math-adjacent area?
- Any personal facts you would like to share. For example, I discussed my family, hometown, educational background, hobbies, and pets.
- Any personal circumstances that might impact your performance in our course
- A recent photo of yourself (I ask to help me identify you, but please ignore this one if you are so inclined).
- Title your message `[math-790] Introduction`.

Wednesday We continued our review of MATH 590 topics, with an emphasis on linear transformations and subspaces. We recalled the definition of the kernel and image (range) of a linear transformation, and what it means for a linear transformation to be an isomorphism. We recalled how to turn a linear transformation between finite dimensional vector spaces into a matrix, and also the basic properties of this process.