Writing project
An ideal recap

Notation. Throughout, V is a vector space over the field k, and $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ is a linear transformation. The zero vector of V is $\mathbf{0} = \mathbf{0}_{\mathsf{V}}$ and the zero transformation on V is denoted $\mathsf{Z} = \mathsf{Z}_{\mathsf{V}}$. This is simply the operator defined by $\mathsf{Z}(\mathbf{v}) = \mathbf{0}_{\mathsf{V}}$ for every $\mathbf{v} \in \mathsf{V}$, which Cooperstein denotes by $0_{\mathsf{V}\to\mathsf{V}}$. We will also assume that V is finite-dimensional, and that $n = \dim(\mathsf{V})$.

Polynomial operators.

(1) Consider polynomials $p = p(x), q = q(x)$ in $\mathsf{k}[x]$. *Briefly*, explain the meaning of the operators $p(\mathsf{T}) : \mathsf{V} \to \mathsf{V}, q(\mathsf{T}) : \mathsf{V} \to \mathsf{V}$, and also why $p(\mathsf{T})q(\mathsf{T}), q(\mathsf{T})p(\mathsf{T}), (pq)(\mathsf{T}), (qp)(\mathsf{T})$ all coincide (recall that, here, multiplication of operators is simply function composition).

Ideals.

> **Definition**: A subset I of $\mathsf{k}[x]$ is an *ideal of* $\mathsf{k}[x]$ if the following conditions are satisfied.
> - (Closure under sums) If $f, g \in I$, then $f + g \in I$.
> - (Absorbing property) If $f \in I$ and $g \in \mathsf{k}[x]$ then $fg = gf \in I$.

(2) Fix $p \in \mathsf{k}[x]$ and set $I = \{pq : q \in \mathsf{k}[x]\}$, the set of all polynomial multiples of p. Prove that I is an ideal. We call I *the ideal generated by* p and call p a *generator for* I.

(3) Prove that $I = \{f \in \mathbb{R}[x] : f(\pi) = 0\}$ is an ideal of $\mathbb{R}[x]$, and that there exists a polynomial $p \in I$ such that I is the ideal generated by p, i.e., all elements in I are multiples of p.

(4) Prove that every ideal I of $\mathsf{k}[x]$ has a generator. **Hint**: This is trivial if $I = \{0\}$ (why?) so let us assume that $I \neq \{0\}$. Let $p \in I$ be a nonzero polynomial with the smallest possible degree. Take an arbitrary polynomial $f \in I$, divide it by p, and solve for the remainder.

(5) Explain why, most of the time, an ideal has infinitely many generators. If I is the ideal generated by p and also the ideal generated by q, then how must p and q be related? Explain why every ideal has a unique *monic* generator.

Minimal polynomials.

(6) Fix a vector $\mathbf{v} \in \mathsf{V}$. Prove that $I_{\mathsf{T}}(\mathbf{v}) = \{f \in \mathsf{k}[x] : f(\mathsf{T})(\mathbf{v}) = 0\}$ is an ideal.
Note: You are being asked to verify the conditions in the definition of an ideal.

(7) Prove that $I_{\mathsf{T}}(\mathbf{v})$ contains a nonzero polynomial of degree at most n.
Hint: Consider the $n + 1$ vectors $\mathbf{v}, \mathsf{T}(\mathbf{v}), \ldots, \mathsf{T}^n(\mathbf{v})$ in the n-dimensional vector space V.

(8) Review the definition of the minimal polynomial of T with respect to $\mathbf{v}$, denoted by Cooperstein as $\mu_{\mathsf{T},\mathbf{v}}(x) \in \mathsf{k}[x]$. Justify Remark 4.1 of Cooperstein.

(9) Solve Problem 4.1: 4 in Cooperstein.

(10) Prove that $I_{\mathsf{T}} = \{f \in \mathsf{k}[x] : f(\mathsf{T}) = \mathsf{Z}_{\mathsf{V}}\}$ is an ideal that contains a nonzero polynomial of degree at most n^2. Review the definition of the minimal polynomial of T, denoted by Cooperstein as $\mu_{\mathsf{T}}(x) \in \mathsf{k}[x]$. Justify Remarks 4.4 and 4.5 of Cooperstein. **Hint**: Use the fact that the vector space $\mathcal{L}(\mathsf{V})$ has dimension n^2, and mimic your previous arguments.

Cyclic subspaces. Fix a vector $\mathbf{v} \in \mathsf{V}$.

> **Definition**: The T-*cyclic subspace generated by* $\mathbf{v}$
> $$\langle \mathsf{T}, \mathbf{v} \rangle = \operatorname{span}(\mathbf{v}, \mathsf{T}(\mathbf{v}), \mathsf{T}^2(\mathbf{v}), \mathsf{T}^3(\mathbf{v}), \ldots) = \{p(\mathsf{T})(\mathbf{v}) : p(x) \in \mathsf{k}[x]\}.$$

(11) Briefly justify the second equality in the above definition.

(12) Solve Problem 4.2: 1 in Cooperstein.

(13) Set $\mathsf{W} = \langle \mathsf{T}, \mathbf{v} \rangle$. Prove that if $\mu_{\mathsf{T},\mathbf{v}}(x)$ has degree d, then $\mathbf{v}, \mathsf{T}(\mathbf{v}), \ldots, \mathsf{T}^{d-1}(\mathbf{v})$ is a basis for W. **Note**: You must show that this list generates W and is linearly independent.

(14) Prove that if $\mathsf{V} = \langle \mathsf{T}, \mathbf{v} \rangle$, then $\mu_{\mathsf{T},\mathbf{v}}(x) = \mu_{\mathsf{T}}(x)$. **Hint**: Earlier, you justified Cooperstein 4.5, and so you know that $\mu_{\mathsf{T},\mathbf{v}}$ divides μ_{T}. To finish, you must show that $\mu_{\mathsf{T},\mathbf{v}}(\mathsf{T}) = \mathsf{Z}_{\mathsf{V}}$ (why?). To do this, use the assumption that $\mathsf{V} = \langle \mathsf{T}, \mathbf{v} \rangle$ and the first problem above.

(15) MORE TO COME.