

Week 10 Tuesday 10/27 and Thursday 10/29

Read Carefully review your notes

Exercises Carefully review your notes

Tuesday We started with Quiz 05, and then presented a solution. We recalled what it means for an operator to be diagonalizable. We proved the **Theorem**: $T : V \rightarrow V$ is diagonalizable if and only if $\mu_T(x)$ is a product of linear polynomials, with no repeated factor. We motivated the Jordan Canonical Form, and recalled the definition of a generalized eigenvector. We proved the **Lemma**: If $S : V \rightarrow V$ is linear, then we have an ascending chain $\ker(S) \subseteq \ker(S^2) \subseteq \cdots$. If any of these containments is an equality, then so are all subsequent containments. In particular, this chain must eventually stabilize at $\ker(S^{\dim(V)})$. We will apply this with the operator $S = T - \lambda I$.

Thursday

Week 09 Tuesday 10/20 and Thursday 10/22

Read Review your notes

Exercises Review your notes

Tuesday Fall Break!

Thursday We recalled the definition of eigenvector and eigenvalue. We then carefully proved that the definition of the characteristic polynomial of an operator T is independent of the choice of the matrix representation of T . Moving on, we proved a lemma that realized a given polynomial in x as the determinant of a carefully defined matrix. We then stated the Cayley-Hamilton Theorem, and went over an incorrect proof, taking care to point out the failure in logic. After this, we presented the correct proof. Our strategy was to prove that if $T : V \rightarrow V$ is linear, and $\mathbf{v} \in V$ is arbitrary, then the little minimal polynomial $\mu_{T,\mathbf{v}}(x)$ divides the characteristic polynomial $\chi_T(x)$.

Week 08 Tuesday 10/13 and Thursday 10/15

Read Cooperstein: 2.4 (review). Review your notes.

Exercises 2.4: 5 (what Cooperstein calls $\mathcal{M}_T(\mathcal{S}, \mathcal{S})$ we called $[T]_{\mathcal{S}}^{\mathcal{S}}.$)

Tuesday We generalized a problem from our (uncollected) group quiz, and Midterm 1, to prove the **Theorem**: If $\mu_T(x) = p_1(x) \cdots p_N(x)$ and the pairs $p_i(x), p_j(x)$ are coprime for all $i \neq j$, then $V = \ker p_1(T) \oplus \cdots \oplus \ker p_N(T)$. We recalled the Fundamental Theorem of Algebra, which says that the only non-constant irreducible polynomials over $\mathbb{C}$ are the irreducible ones. We talked about what this means in the context of our generalization, and motivated by this, recalled the definition of eigenvalues, eigenvectors, and defined a *generalized eigenvector*. We proved that $\mu_T(\lambda) = 0$ if and only if λ is an eigenvalue, and then recalled the definition of the *characteristic polynomial* of a square matrix A , which we denoted by $\chi_A(x)$. We proved that the roots of $\chi_A(x)$ are precisely the eigenvalues of A , and stated (but did not prove) the Cayley-Hamilton Theorem.

Thursday Deviating from usual practice, we discussed Midterm 1 at the start of class (as opposed to the end). We went over averages and common mistakes. We pivoted to math, and recalled the basics of transforming a linear operator into a matrix using bases. We emphasized change-of-coordinate matrices, and proved that matrices A, B represent the same linear transformation if and only if $A = PBP^{-1}$ with P invertible (in this case, P and P^{-1} are appropriate change of basis matrices). We used this to define the characteristic polynomial of a linear transformation, and show that this definition is well-defined.

Week 07 Tuesday 10/06 and Thursday 10/08

Read 4.3

Exercises 4.3: 5, Study for Midterm 1

Tuesday We reviewed our proof of the existence of a maximal vector, emphasizing the algorithmic nature. We considered the notion of T -invariant complementary subspaces, and spent the rest of lecture on a group quiz that was not collected.

Thursday Today was Midterm 1

Week 06 Tuesday 09/29 and Thursday 10/01

Read 4.2, 4.3

Exercises 4.2: 8, 12 (especially Math Ph.D. students), 4.3: 1, 2

Tuesday We proved that the functions Φ and Ψ defined last lecture are inverses, and used this to solve a homework problem. We recalled the meanings of the statements $V = U + W$ and $V = U \oplus W$ for subspaces U, W of V , and proved that $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \{0\}$. We stated a theorem that relates the cyclic subspaces $\langle T, u + w \rangle, \langle T, u \rangle, \langle T, w \rangle$ and polynomials $\mu_{T, u+w}(x), \mu_{T, u}(x), \mu_{T, w}(x)$ whenever the last two are relatively prime.

Thursday We carefully proved the **Theorem**: Suppose that $\mu_{T, u}(x), \mu_{T, w}(x)$ are coprime. Then $\langle T, u + w \rangle = \langle T, u \rangle \oplus \langle T, w \rangle$ and $\mu_{T, u+w} = \mu_{T, u}(x)\mu_{T, w}(x)$. We used this, and the earlier fact that every factor of a little minimal polynomial is a little minimal polynomial, to prove the **Theorem**: There exists $v \in V$ with $\mu_T(x) = \mu_{T, v}(x)$. Our proof was algorithmic. Such a vector v is called *maximal*.

Week 05 Tuesday 09/22 and Thursday 09/24

Read 4.1, 4.2

Exercises 4.1: 6, 8, 9, 12, 4.2: 3, 5, Math Ph.D. students (or interested undergrads): Prove that Φ and Ψ are inverses.

Tuesday We reviewed the group quiz, and carefully proved that $\dim \langle T, v \rangle = \deg \mu_{T, v}(x)$. We defined the (big) minimal polynomial of an operator, and proved that it exists and has degree at most $\dim(V)^2$, a bound we will soon improve. We proved that every $\mu_{T, v}(x)$ divides $\mu_T(x)$, justifying our choice of terminology of small minimal polynomial for $\mu_{T, v}(x)$ and big minimal polynomial for $\mu_T(x)$.

Thursday We took Quiz 03. We proved that if $\langle T, v \rangle = V$, then $\mu_T(x) = \mu_{T, v}(x)$. We made the **Definition**: If $\langle T, v \rangle = V$, and W is a T -invariant subspace of V , then $I_W = \{f(x) \in \mathbb{F}[x] : f(T)(v) \in W\}$. We proved that I_W is ideal that contains $\mu_T(x) = \mu_{T, v}(x)$, and then stated **Theorem**: If $\langle T, v \rangle = V$, then there is a bijection $\{T\text{-invariant subspaces } W \text{ of } V\} \leftrightarrow \{\text{monic factors } p(x) \text{ of } \mu_T(x)\}$. We defined $\Phi(W)$ to be the monic generator of the ideal I_W and $\Psi(p(x))$ to be the T -invariant subspace $\langle T, p(T)(v) \rangle$. We proved that a monic polynomial has finitely many monic divisors. As a consequence, we saw that if T is cyclic, this bijection tells us how to count and describe all T -invariant subspaces.

Week 04 Tuesday 09/15 and Thursday 09/17

Read 4.1, 4.2

Exercises Complete the group quiz

Tuesday We worked on a group quiz. The topics were polynomials, minimal polynomials of operators, and cyclic subspaces. We will continue working on this on Thursday.

Thursday We worked on the group quiz all lecture.

Week 03 Tuesday 09/08 and Thursday 09/10

Read 3.1, 4.1

Exercises 3.1: 1 (assume the field is $\mathbb{Q}$), 5, 8, 10, 4.1: 3, 4

Tuesday We defined the terms *relatively prime* and *irreducible* for polynomials. We applied Bezout's Identity to demonstrate that irreducible polynomials behave much like prime integers. We stated, and mostly proved, the Unique Factorization Theorem for polynomials over a field. We then discussed the connection between roots of a polynomial and linear factors. After this, we returned to linear algebra, and recalled how to substitute an operator for the variable in a polynomial to obtain an operator. We ended with an example of an operator $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and a nonzero polynomial $f(x)$ such that $f(T)$ is the zero operator.

Thursday We took Quiz 02. We motivated and proved the **Theorem**: If $T : V \rightarrow V$ is an operator on a finite dimensional vector space V , then there exists a polynomial $0 \neq p(x) \in \mathbb{F}[x]$ such that $p(T)(\mathbf{v}) = \mathbf{0}$. In fact, there exists a least such polynomial, that is, a monic polynomial $\mu_{T,\mathbf{v}}(x) \in \mathbb{F}[x]$ such that $\mu_{T,\mathbf{v}}(T)(\mathbf{v}) = \mathbf{0}$ and also such that $\mu_{T,\mathbf{v}}(x)$ divides any other polynomial $p(x) \in \mathbb{F}[x]$ with $p(T)(\mathbf{v}) = \mathbf{0}$. This is called the minimal polynomial of T with respect to $\mathbf{v}$. We then carefully went over an example of how to compute this using 290 methods.

Week 02 Tuesday 09/01 and Thursday 09/03

Read 3.1, 3.2, 4.1

Exercises 3.1: 1 (assume the field is $\mathbb{Q}$)

Tuesday We continued our discussion of linear transformations and matrices. We defined the vector space $\mathcal{L}(V, W)$ and explained why $\mathcal{L}(V) = \mathcal{L}(V, V)$ is special. We stated the existence of an isomorphism between $\mathcal{L}(V, W)$ and the vector space of $\dim(V) \times \dim(W)$ matrices, when both of these dimensions are finite. The rest of lecture was dedicated to polynomials. We discussed how to substitute a linear transformation for a variable in a polynomial. After introducing notation, and examples, we proved the Division Algorithm for polynomials over a field.

Thursday We defined what it means for a polynomial to divide another, and recalled the definition of a greatest common divisor of two polynomials (not both zero). We then recalled the Euclidean Algorithm for computing a greatest common divisor, and covered an example. There, we saw that the greatest common divisor of a pair of polynomials in $\mathbb{F}[x]$ can be written as an $\mathbb{F}[x]$ -linear combination of those polynomials. This provided an example of the Bezout Identity

Week 01 Tuesday 08/25 and Thursday 08/27

Read Review 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 2.1, 2.2, 2.4, 2.5, 2.6

Exercises Review the above sections of Cooperstein

Tuesday Welcome to Math 790! We presented the syllabus and a basic introduction. We then began our review of some basic facts, i.e., material from MATH 590. We recalled the definition of an abstract vector space, presented examples, recalled the concepts of linear independence, span, and bases. We discussed infinite dimensional vector spaces, though we focus on the finite dimensional case.

Thursday We continued our review of 590 topics, emphasizing linear transformations and subspaces. We recalled the definition of the kernel and image (range) of a linear transformation, and what it means for a linear transformation to be an isomorphism. We recalled how to turn a linear transformation between finite dimensional spaces into a matrix, and the basic properties of this process.