Hernández Lecture 3

Friday, June 10, 2022 9:06 AM

<u>Last time</u>: $R = \mathbb{F}_p[x_1, \dots, x_n]$ $\quad \mathfrak{m} = \langle x_1, \dots, x_n \rangle$

$F: R \to R$ def. by $g \mapsto g^p$ Frobenius

"breaks up" $R \overset{F}{\cong} F(R) \subseteq R$ makes F onto (already injective)

$F(R) = R^p = \{g^p : g \in R\} = \mathbb{F}_p[x_1^p, \dots, x_n^p]$

$\cdots \subseteq R^{p^e} \subseteq \cdots \subseteq R^{p^2} \subseteq R^p \subseteq R \subseteq \underbrace{R^{1/p} \subseteq \cdots}$ $=:$ FF (Frobenius fractal)

$R^{p^2} = (R^p)^p$

$\cdots \subseteq S^{p^{e-1}} \subseteq \cdots \subseteq S^p \subseteq S$ $\quad$ ($R^{p^e} = S^{p^{e-1}}$, $R^{p^2} = S^p$, $R^p = S$)

need char p to go this direction

never terminates in both directions

OK w/ Noetherian b/c not ideals (they are subrings)

Each possible extension is finite & free

Every way to move in FF is a ring homomorphism

- "move up" = inclusion = ring map! can still do this in char 0
- "move down" = "iterates of F" Need char p for this

"<u>Frobenius powers of ideals</u>" (uses FF to construct different canonical ideals)

☹ If $\varphi: A \to B$ map of rings, $I \subseteq A$ ideal, then $\varphi(I)$ need not be an ideal of B

e.g. $\mathbb{Z} \hookrightarrow \mathbb{Q}$ or $k \hookrightarrow k[x]$

So we are forced to take $\langle \varphi(I) \rangle \subseteq B$ ← now this is an ideal

When is $\varphi(I)$ an ideal? When φ is onto

When is $\varphi(I)$ an ideal? When φ is onto

$$\begin{array}{ccc} R^p_{=B} & \subseteq & R_{=A} \\ \text{ideal}\ \cup\!\mathsf{I} & & \cup\!\mathsf{I}\ \text{ideal} \\ F(I) & \xleftarrow[F \text{ onto}]{} & I \\ \| & & \\ \{g^p : g \in I\} & & \end{array}$$

$\{g^p : g \in I\} \overset{\text{inclusion}}{\subseteq} \langle F(I) \rangle =: I^{[p]}$, where $\langle F(I)\rangle = \langle g^p : g \in I \rangle$ — Frobenius p^{th} power of I

$F(I)$ not an ideal of R (e.g. $x \cdot x^p \notin F(I)$), so consider $\langle F(I) \rangle$

How is I^p different from $I^{[p]}$?

Ex: If $I = \mathfrak{m}$

$$\mathfrak{m}^{[p]} = \langle x_1^p, \dots, x_n^p \rangle$$
$$\cap\!\mathsf{I}$$
$$\mathfrak{m}^p = \langle x_1^{a_1} \cdots x_n^{a_n} \mid a_1 + \cdots + a_n = p \rangle$$

$$I^{[p^e]} := \langle g^{p^e} : g \in I \rangle$$

Exercise: $\dfrac{1}{\operatorname{mult}(f)} = \sup \left\{ \dfrac{t}{q} \mid f^t \notin \mathfrak{m}^q \right\}$

Now consider Frobenius version:

$$FIS(f) := \sup \left\{ \frac{t}{p^e} : f^t \notin \mathfrak{m}^{[p^e]} \right\}$$

↑ Frobenius index of singularities

or "F-pure threshold"

Exercise: $f^t \notin \mathfrak{m}^q$ depends on rep. of $\lambda = \dfrac{t}{q}$, however

$f^t \notin \mathfrak{m}^{[p^e]}$ is independent of $\lambda = \dfrac{t}{p^e} = \dfrac{tp}{p^{e+1}}$ ← advantage

$g^p \in I^{[p]} \Leftrightarrow g \in I$ (our R)

$g' \in I^{[p]} \iff g \in I$ (our R)

Facts: • $\mathrm{FIS}(f) \in [0,1] \cap \mathbb{Q}$

issue b/c of sup so not obvious its in $\mathbb{Q}$.

• 1 if f is nonsingular (at $\underline{0}$)

• smaller values $\longleftrightarrow$ "more sings"

No issue w/ Noetheriarity:

$$\mathfrak{m}^{[p^2]} \subseteq \mathfrak{m}^{[p]} \subseteq \mathfrak{m}$$

$$\langle \underline{x}^{p^2} \rangle = (x^p)^p \qquad \langle \underline{x}^p \rangle$$

Exercise: $f^t \notin \mathfrak{m}^{[p^e]}$ is very meaningful in terms of FF

$$\underbrace{R^{p^e} \subseteq \cdots \subseteq R}$$

can actually ignore immediate inclusions & just consider:

$$\text{constants} \rightarrow R^{p^e} \subseteq R \leftarrow \text{vectors}$$

finite & free (so we can mimic linear algebra)

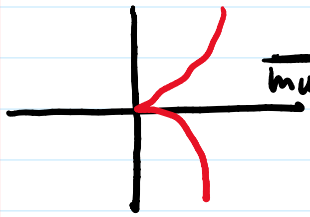
"basis" $= \{x_1^{a_1} \cdots x_n^{a_n} : \text{all } 0 \le a_i < p^e\}$

So Exercises $\Rightarrow f^t \notin \mathfrak{m}^{[p^e]} \iff \exists$ a basis for R over R^{p^e} containing f^t

So f^{p^e} can NEVER be part of a basis

Exercise: $\mathrm{FIS}(y^2 - x^3) = \begin{cases} 1/2 & p=2 \\ 2/3 & p=3 \\ 5/6 & p \equiv 1 \pmod 6 \end{cases}$

[illegible]



$\frac{1}{\text{mult}(f)} = \frac{1}{2}$

$$\begin{cases} 2/3 & p = 3 \\ 5/6 & p \equiv 1 \pmod 6 \\ 5/6 - 1/(6p) & p \equiv 5 \pmod 6 \end{cases}$$

only options b/c primes are units

both happen about 50% of the time

$\lim_{p \to \infty} FIS(f_p) =$ exists   very deep

$=: AIS(f) =$ analytic index of sings $= \{\lambda : \frac{1}{|f|^\lambda} \text{ is } L^2 \text{ at } \underline{0}\}$

complex norm

square integrable

$=$ log canonical threshold

We can visualize FF as fractals pictorially & use geometry of fractals to make results

& the shape fills up & is limit $\lim_{p \to \infty} FIS(f_p)$ (& this limit must exist)