Last time: $R = \mathbb{F}_p[x_1, \ldots, x_n]$ $\eta = \langle x_1, \ldots, x_n \rangle$

$F: R \to R$ def. by $g \to g^p$, Frobenius

"breaks up" $R = \mathbb{F}_p$ $F(R) = R$ makes $F$ onto (already injective)

$R^p = \{ \sum g_i^p : g_i \in R \} = \mathbb{F}_p[x_1^p, \ldots, x_n^p]$

\[
\ldots \leq R^p \leq \ldots \leq R^{2p} \leq R^p \leq R \leq R^{1p} \leq \ldots = : FF \text{ (Frobenius fractal)}
\]

\[
\begin{align*}
\ldots & \leq R^{p^{n-1}} \leq R^p \leq \ldots \leq R^p \leq \ldots \\
& \leq S^p \leq \ldots \leq \mathbb{F}_p = S
\end{align*}
\]

\[\text{need charp to go this direction}\]

\[\text{never terminates in both directions}\]

Each possible extension is finite & free

Every way to move in $FF$ is a ring homomorphism

- "move up" = inclusion = ring map! can still do this in char 0
- "move down" = "iterates of $F" \text{ Need charp for this}

"Frobenius powers of ideals" (uses $FF$ to construct different canonical ideals)

If $\Psi: A \to B$ map of rings, $I \subseteq A$ ideal, then $\Psi(I)$ need not be an ideal of $B$

\[\text{e.g. } \mathbb{Z} \rightarrow \mathbb{Q} \text{ or } k \rightarrow k[x] \]

\[\text{now this is an ideal} \]

So we are forced to take $\langle \Psi(I) \rangle \subseteq B$

When is $\Psi(I)$ an ideal? When $\Psi$ is onto
When is $\mathfrak{F}(I)$ an ideal? When $\mathfrak{F}$ is onto

$$\mathfrak{F}^p \subseteq \mathfrak{F} =: A$$

and $\mathfrak{I}$ is ideal

$$\mathfrak{I} \subseteq F(I) =: \mathfrak{F}^p \subseteq \mathfrak{F}$$

for $p$th Frobenius power of $I$

$F(I)$ is not an ideal of $R$ (e.g. $x \cdot x^p \notin F(I)$), so consider $\langle F(I) \rangle$

How is $\mathfrak{I}^p$ different from $\mathfrak{I}^p$?

**Example:** If $I = \mathfrak{m}$

$$\mathfrak{m}^p = \langle x_1^p, \ldots, x_n^p \rangle$$

$$\mathfrak{m}^p = \langle x_1^p \cdots x_n^p \mid a_1 + \cdots + a_n = p \rangle$$

$$\mathfrak{I}^p := \langle g^p : g \in \mathfrak{I} \rangle$$

**Exercise:**

$$\frac{1}{\text{mult}(f)} = \sup \left\{ \frac{t}{q^t} \mid f^t \notin \mathfrak{m}^p \right\}$$

Now consider Frobenius version:

$$\text{FIS}(f) := \sup \left\{ \frac{t}{p^t} : f^t \notin \mathfrak{m}^p \right\}$$

Frobenius index of singularities or “F-pure threshold”

**Exercise:** $f^t \notin \mathfrak{m}^p$ depends on rep. of $\lambda = \frac{t}{q^t}$, however $f^t \notin \mathfrak{m}^p$ is independent of $\lambda = \frac{t}{p^t} = \frac{tp}{p^{pe}}$ — advantage

$$g^p \in \mathfrak{I}^p \iff g \in \mathfrak{I} (\text{our } R)$$
$g \in I^{\text{ur}} \iff g \in I \text{ (our } R)$

**Facts:**
- $FIS(f) \in [0, 1] \cap \mathbb{Q}$
- $\text{issue b/c of sup so not obvious its in } \mathbb{Q}$

- If $f$ is nonsingular (at 0)
- Smaller values $\leftrightarrow$ "more sings"

No issue w/ Noetheriarity:

\[
\eta^{[p^2]} \subseteq \eta^{[R]} \subseteq \eta^{[F]}
\]

\[
\langle x^{p^2} \rangle \subseteq \langle x^p \rangle \subseteq (x^p)^p
\]

**Exercise:** $f^t \notin \eta^{[p^e]}$ is very meaningful in terms of FF

\[
R^{p^e} \subseteq \cdots \subseteq R
\]

- Can actually ignore immediate inclusions &
  just consider:
- Constants $\to R^{p^e} \subseteq R \leftarrow$ vectors

  finite & free (so we can mimic linear algebra)

  "basis" $= \{ x_1^{a_1}, \ldots, x_n^{a_n} : \text{ all } 0 \leq a_i < p^e \}$

So Exercises $\Rightarrow f^t \notin \eta^{[p^e]} \iff \exists \text{ a basis for } R \text{ over } R^{p^e}$

containing $f^t$

So $f^{p^e}$ can NEVER be part of a basis

**Exercise:** $FIS(y^2 - x^3) = \left\{ \begin{array}{ll}
\frac{1}{2} & \text{p = 2} \\
\frac{2}{3} & \text{p = 3} \\
\frac{p - 1}{2p} & \text{p \equiv 1 (mod 4)}
\end{array} \right.$
\[ \frac{1}{\text{mult}(f)} = \frac{1}{2} \cdot \left( \frac{2/3}{5/6} - \frac{5/6}{11(\rho)} \right) \]

\[ p = 3 \quad p = 1 \pmod{6} \quad p = 5 \pmod{6} \]

- only options that fit primes are units.
- both happen about 50% of the time.

\[ \lim_{p \to \infty} FIS(f_p) = \text{exists very deep} \]

\[ =: \text{AIS}(f) = \text{analytic index of singularities} = \left\{ x : \frac{1}{\sqrt[3]{|f|^{1/3}}} = a + 0 \right\} \]

\[ = \log \text{canonical threshold} \]

We can visualize FF as fractals pictorially & use geometry of fractals to make results.

& the shape fills up & is limit \( \lim_{p \to \infty} FIS(f_p) \) (this limit must exist)