BRIDGES 2022

June 7, 2022

Algebraic hypersurfaces (especially singular ones)

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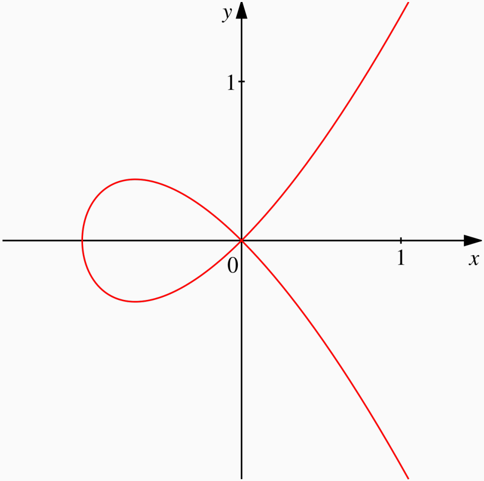
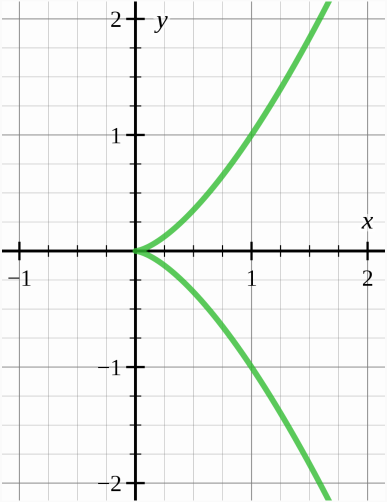
- **Fundamental question**: How complicated can $\mathbb{V}(f)$ be in a neighborhood of a *singular point*?

The node $y^2 - x^3 - x^2 = 0$ when $k = \mathbb{R}$

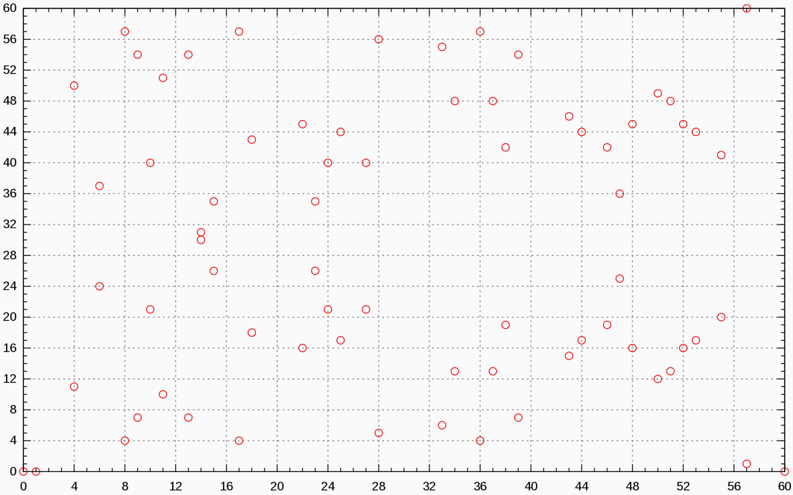


The cusp $y^2 - x^3 = 0$ when $k = \mathbb{R}$



Hauser's Gallery of Singular Algebraic Surfaces

The cusp $y^2 - x^3 = 0$ when $k = \mathbb{Z}/61\mathbb{Z}$



The cusp $y^2 - x^3 = 0$ when $k = \mathbb{Z}/89\mathbb{Z}$