**Course Digest, MATH 291, Fall 2024**

| | |
|---|---|
| **Week 16** | Tuesday 12/10 and Thursday 12/12 |
| **Read** | Work on Honors Exploration, Lyryx assignment |
| **Homework** | Lyryx: Posted on Tuesday<br>Practice: Work on Honors Exploration, Lyryx assignment |
| **Tuesday** | We worked on the last Honors Exploration in class. We will continue doing so on Thursday. |
| **Thursday** | We worked on the last Honors Exploration in class |

| | |
|---|---|
| **Week 15** | Tuesday 12/03 and Thursday 12/05 |
| **Read** | 3.3, 3.4 |
| **Homework** | Lyryx: Posted<br>Practice: 3.3.1 |
| **Tuesday** | We started by recalling the definitions of eigenvalues and eigenvectors associated to a square matrix. We also recalled/reproved the algorithm for computing these objects, and spent the rest of lecture going over examples. |
| **Thursday** | We worked on the last Honors Exploration in class. We will continue doing so on Tuesday. |

**Week 13** Tuesday 11/13 and Thursday 11/15

**Read** Enjoy the break

**Homework** Lyryx: None
Practice: Enjoy the break

**Tuesday** Our entire discussion was motivated by the observation: If $A$ is an $n \times n$ matrix, then if $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then so is $A\mathbf{x}$, and it makes sense to ask how they compare with one another. A first question in this direction is **Question**: When is $A\mathbf{x}$ a constant multiple of $\mathbf{x}$, i.e., when is $A\mathbf{x}$ a stretched version of $\mathbf{x}$? After considering some examples, we noted that this was always true if $\mathbf{x} = \mathbf{0}$, and so excluding this, we arrived at the following **Definition**: If $A$ is a square matrix, then a vector $\mathbf{x}$ is an *eigenvector* for $A$ if (1) $\mathbf{x}$ is not the zero vector, and (2)$A\mathbf{x} = \lambda\mathbf{x}$ for some constant $\lambda$. In this case, we call $\lambda$ the *eigenvalue* corresponding to the eigenvector $\mathbf{x}$. After going over more examples, we derived the **Algorithm**: To determine all possible eigenvalues, find all parameters $\lambda$ such that $\det(A - \lambda I) = 0$. At the end of lecture, we returned Midterm 02, and discussed the scale.

**Thursday** Happy Thanksgiving!

**Week 13** Tuesday 11/19 and Thursday 11/21

**Read** 3.2

**Homework** Lyryx: Posted on Lyryx, due Thursday morning
Practice: 3.2.2, 3.2.5, 3.2.10abcdef

**Tuesday** We started class by going over Quiz 06. After that, we proved that we can add the 5th point in the following **Theorem**: If $A$ is a square matrix, then the following statements are equivalent.

(1) $A$ is invertible.
(2) The only solution to the linear system $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
(3) The reduced row echelon form (RREF) of $A$ is $I$, the identity.
(4) $A$ is the product of elementary matrices.
(5) $\det(A) \neq 0$.

We spent the rest of the lecture discussing how to use this, along with our knowledge of properties of the determinant.

**Thursday** Today is Midterm 02.

**Week 12** Tuesday 11/12 and Thursday 11/14

**Read** Finish 3.1, start 3.2

**Homework**
Lyryx: Posted on Lyryx, updated Thursday
Practice: 3.1.5, 3.1.6, 3.1.7, 3.1.9, 3.1.13, 3.2.3, 3.2.4, 3.2.5
Meditate on the beautiful formula $\det(AB) = \det(A)\det(B)$.

**Tuesday** Building off the last lecture, we started with the **Theorem**: When computing the determinant of a square matrix, you can expand across any row or across any column, provided that you respect the $\pm$ pattern. After going over some examples of this, we asked the following **Question**: How does the determinant change when we perform a single row operation? More precisely, if we have $A \to B$ is a single row operation, then how are $\det(B)$ and $\det(A)$ related? We worked out the $2 \times 2$ case, and motivated by this, we presented the following **Theorem**: Suppose that $A$ is a square matrix, of arbitrary size, and that $B$ can be obtained from $A$ by performing a single elementary row operation. Then $\det(B) = k\det(A)$, where $k$ is as follows:

| Row operation | Value of $k$ |
|---|---|
| $R_i \to R_i + \lambda R_j$ | 1 |
| $R_i \leftrightarrow R_j$ | -1 |
| $R_i \to \lambda R_i, \lambda \neq 0$ | $\lambda$ |

We ended lecture by going over some examples of how to use this theorem. We also discussed how to use it iteratively when simplifying $A \to B_1 \to \cdots B_N = B$ using many row operations: If $k_i$ is the constant determined by the simplification that terminates at $B_i$, then $\det(B) = k_N \cdots k_2 k_1 \det(A)$.

**Thursday** We discussed further properties of the determinant that follow from the main results from the previous lecture. More argued that if $A$ is an $n \times n$ matrix, then $\det(A) = \det(A^T)$, where $A^T$ stands for the transpose of $A$, and that if $k$ is a constant, then $\det(kA) = k^n A$. After this, we stated the following **Beautiful Theorem**: If $A, B$ are square of the same size, then $\det(AB) = \det(A)\det(B)$. In words, the determinant of a product is the product of the determinants. We discussed how to use this iteratively to simplify the determinant of the product of an arbitrary number of matrices (as opposed to just two) and also how to use this to simplify the determinant of a matrix raised to a power. We also showed how it implies the following **Theorem**: If $A$ is invertible, then $\det(A)$ is nonzero, and $\det(A^{-1}) = \det(A)^{-1}$. In words, the determinant of an inverse is the inverse of the determinant.

**Week 10** Tuesday 10/23 and Thursday 10/25

**Read** 3.1

**Homework** Lyryx: Posted Thursday
Practice: 3.1.1abcde

**Tuesday** We started with a quick quiz. After that, we stated, and finally proved, the following **Theorem**: If $A$ is a square matrix, then the following statements are equivalent.

(1) $A$ is invertible.
(2) The only solution to the linear system $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
(3) The reduced row echelon form (RREF) of $A$ is $I$, the identity.
(4) $A$ is the product of elementary matrices.

After the proof, we expressed a desire to add a fifth item to the list, namely, that the *determinant* of $A$ is nonzero. We also discussed what a determinant was when $A$ was small, i.e., $1 \times 1$ or $2 \times 2$. To try to establish a pattern, we introduced the following **Definition**: If $A$ is a square $n \times n$ matrix, then $A_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting the $i$-th row and $j$-th column of $A$.

**Thursday** We recalled the key ideas from the previous lecture, and were able to identify a plausible pattern to follow for the formula of the determinant of a $3 \times 3$ matrix. Motivated by this, we introduced the following recursive (inductive) **Definition**: If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n1} \\ a_{21} & a_{22} & \cdots & a_{21} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

then $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots \pm a_{1n}\det(A_{1n})$. We discussed the way to determine the $\pm$ signs in this, and went over examples. We also concluded by proving that, in the $2 \times 2$ case, you can compute determinants by "expanding" using any row or column, provided that you get the $\pm$ signs correct. We asserted that this is true for square matrices of any size, but provided no justification.

e then stated, and proved, the following **Theorem**: If $A$ is a square matrix, then the following statements are equivalent.

**Week 10** Tuesday 10/29 and Thursday 10/31

**Read** 2.5, 3.1

**Homework** Lyryx: Posted, updated Thursday

Practice: 2.5.6, 2.5.7, 2.5.8, 2.5.15, 2.5.20.

Problems in red are challenge problems, i.e., just for fun!

**Tuesday** We started off by recalling the basics of elementary matrices. We then saw that it is easy to compute powers of elementary matrices, and products of powers of elementary matrices.

After this, we proved the following **Theorem**: If $A$ is invertible, then the RREF of $A$ must be the identity. We also explained how to use this result to factor an invertible matrix $A$ into a product of elementary matrices.

**Thursday** Happy Halloween! Today, we started class by recalling how to factor a matrix. We first explained how this process of factoring explains our previously mysterious algorithm $[A|I] \sim \cdots \sim [I|A^{-1}]$ for computing the inverse of an invertible matrix. After this, we went over some examples of how to factor an invertible matrix into a product of elementary matrices. We then recalled some basic notions from logic. We defined what we mean by a *logical statement*, and discussed the difference between an *implication* and an *equivalence*, going over many examples, both from math and from the real world. With this as motivation, we then stated the following **Theorem**: If $A$ is a square matrix, then the following statements are equivalent.

(1) $A$ is invertible.

(2) The only solution to the linear system $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

(3) The reduced row echelon form (RREF) of $A$ is $I$, the identity.

(4) $A$ is the product of elementary matrices.

We will prove this next week.

**Week 09** Tuesday 10/22 and Thursday 10/24

**Read** 2.4, 2.5

**Homework** Lyryx: Posted, updated Thursday
Practice: 2.4.4, 2.4.5, 2.4.6, 2.5.1, 2.5.2

**Tuesday** We first focused on the Cancellation Property for real numbers (our usual constants). Motivated by this, we proved the **Cancellation Property for Invertible Matrices**: If $A$ is invertible, and $AB = AC$, then $B = C$. Similarly, if $BA = CA$, then $B = C$. We also presented examples that illustrate that this can be false when $A$ is not invertible. Next, we presented a list of basic properties of inverses, e.g., we explained why the product of any number of invertible matrices is invertible, and gave a formula for the inverse. We then used these basic properties to solve a simple equation involving matrices and inverses.

After this, we introduced the concept of an elementary matrix. Recall that a matrix $E$ is elementary if it can be obtained from the identity matrix $I$ by performing a single elementary row operation. We went over many examples to illustrate this definition, as well as some non-examples. After this, we presented the following **Theorem**: Suppose that $B$ can be obtained from $A$ by a single row operation $A \to B$. If $E$ is the elementary matrix corresponding to this same row operation (i.e., the elementary matrix obtained by performing this same row operation on the identity matrix $I$), then $B = EA$. We went over a simple example of this before ending the lecture.

**Thursday** We started our class period with Quiz 04. We then started the lecture with **Theorem**: Suppose that $B$ can be obtained from $A$ by a single row operation $A \to B$. If $E$ is the elementary matrix corresponding to this same row operation (i.e., the elementary matrix obtained by performing this same row operation on the identity matrix $I$), then $B = EA$. We went over a number of examples to illustrate this. We emphasized that the $A$ and $B$ in this theorem need not be square matrices (though $E$ must be).

Next, discussed the **Theorem**: If $E$ is elementary, then $E$ is invertible. In fact, $E^{-1}$ is also elementary: If $E$ corresponds to a particular row operation, then $E^{-1}$ corresponds to the *inverse (or undoing)* row operation. We then went over many examples to illustrate this very useful fact.

We ended by talking about what happens when we perform a string (i.e., more than one) elementary row operations, arriving at the following: If $A \to B_1 \to B_2 \to B_3 \cdots \to B_n = B$, and the operation to get from to $B_i$ from the previous matrix corresponds to the elementary matrix $E_i$, then $B = E_n E_{n-1} \cdots E_3 E_2 E_1 A$.

**Week 08** Tuesday 10/15 and Thursday 10/17

**Read** 2.4

**Homework** Lyryx: Posted, updated Thursday
Practice: 2.4.1, 2.4.2

**Tuesday** Fall Break 2024!

**Thursday** We recalled the key ideas from last lecture, and then presented the following algorithm for determining whether a square matrix was invertible. **Algorithm**: Let $A$ be a square matrix, say of size $n \times n$, and let $I$ be the $n \times n$ identity matrix. If it is possible to apply row operations to obtain $[A|I] \to \cdots \to [I|B]$ then $A$ is invertible, and $B = A^{-1}$. If this is not possible (e.g., if it is not possible to transform $A$ into $I$, i.e., if the RREF of $A$ is not $I$), then $A$ is not invertible. We went over examples of this, and then presented the following special case for $2 \times 2$ matrices. **Theorem** Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then $A$ is invertible, and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = 0$, then $A$ is not invertible.

**Week 07** Tuesday 10/08 and Thursday 10/10

**Read** 2.4

**Homework** Lyryx: None this week
Practice:

**Tuesday** Today was Midterm 01.

**Thursday** We continued our discussion of invertible matrices. We recalled the definition and some interesting examples from last time. After this, we presented examples (with proof) of matrices that had no inverse. We then proved the following **Theorem**: If $A$ is invertible, then it has a unique inverse. In other words, if $B, C$ are two inverses for $A$, then $B = C$. This justifies our use of the notation $A^{-1}$ to denote the inverse of $A$. The last portion of class was spent dealing with the fallout from Midterm 01 and Exploration 01.

**Week 06** Tuesday 10/01 and Thursday 10/03

**Read** 2.3

**Homework** Lyryx: Posted, updated Thursday
Practice: 2.3.1, 2.3.2, 2.3.3

**Tuesday** Today, we built on our matrix $\times$ column multiplication to define matrix $\times$ matrix multiplication. To motivate this, we defined a *linear transformation* to be a function $f : \mathbb{R}^n \to \mathbb{R}^m$ of the form $f(x) = Ax$, where $A$ is a fixed $m \times n$ matrix. After going over (non)examples, we considered the following **Question**: If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^p \to \mathbb{R}^n$ are linear transformations associated to matrices $A, B$, then what, if any, matrix is the composition $f \circ g : \mathbb{R}^p \to \mathbb{R}^m$ associated to? After unraveling this, we say that the answer was the matrix whose $i$-th column is obtained by applying $A$ to the $i$-th column of $B$. We then used this to *define* the product of matrices $AB$.

**Thursday** We started lecture by reviewing the definition of matrix multiplication. We then listed the basic properties of matrix multiplication (e.g., associativity, various distributive properties), and even proved a few. We then shifted tracks, and recalled the basic properties of inverses, or reciprocals, of constants, and motivated by this, we drew a series of analogies with matrices. Based on this, we introduced the following **Definition**: A matrix $A$ is *invertible* if there exists a matrix $B$ such that $AB = BA = I$. In this case, we call $B$ the *inverse* of $A$, and we write $B = A^{-1}$. **Note**: As we explained in class, if a matrix is invertible, then it must also be square. We went over a few examples, computing the inverse of some $2 \times 2$ matrices.

**Week 05** Tuesday 09/24 and Thursday 09/26

**Read** 2.1, 2.2

**Homework**
- Lyryx: Posted
- Practice: 2.1.1, 2.1.4, 2.1.8, 2.1.15, 2.2.2, 2.2.3, 2.2.4

**Tuesday** We finally introduced matrices, and our focus today was on matrix algebra. We discussed how to scale a matrix by a constant, and how to add and subtract matrices of like sizes. After this, we recorded the basic properties of matrix scaling, addition, and subtraction (e.g., associativity, and multiple notions of the distributive property). After going over some examples, we defined the transpose of a matrix, and then listed the basic properties of this operation. We concluded lecture by solving more equations involving scaling, adding and subtracting, and also transposing, matrices.

**Thursday** We defined how to multiply an $m \times n$ matrix $A$ by a column vector of size $n$ to get a column of size $m$. In other words, and $m \times n$ matrix times a $n \times 1$ matrix is a $m \times 1$ matrix. We went over numerous examples, and concluded the math-portion of the lecture by describing how every system of linear equations can be described as a *single* matrix equation. We ended the actual lecture with Quiz 03.

**Week 04** Tuesday 09/17 and Thursday 09/19

**Read** Finish 1.3, start 2.1

**Homework**
- Lyryx: Posted, updated Thursday
- Practice: 1.3.2, 1.3.5

**Tuesday** We recalled the following basic fact about homogeneous systems: *Every solution of a homogeneous system is a linear combination of finitely many special solutions, which we call basic solutions.* We also discussed how this meant that an infinite amount of data (the solutions to a homogeneous system with infinitely many solutions) can be described in finite terms (i.e., finitely many basic solutions). We went over examples to compute basic solutions. We noted that multiplying a basic solution by a non-zero constant is also a basic solution. After this, we considered some work of the ancient Greeks, and discussed how we might recover them using linear algebra. As an example, we found the equation for a circle through three points that do not lie on any line.

**Thursday** We started off with Quiz 02. After this, we went over a few examples of how to solve a homogeneous system when the augmented matrix contains an unknown parameter. Our philosophy here was as follows: *Avoid dealing with any unknown parameter until you absolutely must.* This means that you will have to be less aggressive when performing Gaussian elimination.

**Week 03** Tuesday 09/10 and Thursday 09/12

**Read** 1.3

**Homework**

Lyryx: Posted, updated on Thursday

Practice: 1.3.3, 1.3.4, 1.3.5

**Tuesday** We started the class with Quiz 01. Afterwards, we started the lecture portion by considering an example of Guassian Elimination (GE) with parameters. After that, we introduced the terminology of points/vectors/columns, and discussed how to add/subtract these, and also how to scale by a constant. After this, we defined what it means to be a linear combination of two points.

**Thursday** We started off by recalling the definition of a linear combination. After discussing some natural questions, we extended the definition of a linear combination to account for linear combinations of any number of vectors, and then went over an example about how to determine whether a particular point is a linear combination of other given points. The key idea was to translate this question into a linear system, and then solve that system.

After this, we defined a *homogeneous system*, which is merely a linear system with every right-hand side consisting of the number 0. We saw, via an example, that a linear combination of two solutions to a homogeneous system is also a solution, and then presented some Basic facts about homogeneous systems: (0) Every homogeneous system has *trivial solution* obtained by setting all of the unknowns equal to zero. (1) The trichotomy for general systems of linear equations becomes a *dichotomy* for homogeneous systems, i.e., every homogeneous system has a unique solution (the trivial solution) or has infinitely many solutions. (2) A linear combination of solutions to a homogeneous system is a solution. After this, asked whether all solutions to a homogeneous system can be “generated” by only a finite number of solutions, and covered an example to illustrate this.

**Week 02** Tuesday 09/03 and Thursday 09/05

**Read** §1.2

**Homework**
- Lyryx: Posted, due next Monday at midnight.
- Practice: 1.1.1, 1.1.7, 1.1.8, 1.2.1, 1.2.3

**Tuesday** We recalled the main points from the previous lecture, namely, the elementary operations on equations in a linear system. After this, we introduced the *augmented matrix* associated to a linear system, and described the *elementary row operations*. We then used these elementary row operations to solve some systems of equations, and concluded the lecture by asking the following Question: How do we know when to stop simplifying a linear system using elementary row operations?

**Thursday** Attempting to answer the question appearing above, we introduced the concept of *echelon form* and *reduced echelon form*. After going over examples, we presented, but did not prove, the following **Theorem**: By applying elementary row operations, every augmented matrix may be transformed to a *unique* reduced echelon form. After this, we discussed the method of Gaussian Elimination, which is an algorithm for solving a system of linear equations by transforming it to (reduced) echelon form. After this, we spent the rest of the lecture going over examples of this.

**Week 01** Tuesday 08/27 and Thursday 08/29

**Read** §1.1, §1.2

**Homework**
Lyryx: None this week
Practice: None this week

**Tuesday** Welcome to Math 291! The entire period was spent on the syllabus and introductions. We will do math next lecture.

Special assignment: Read our syllabus, and introduce yourself to me via email. Do this ASAP, and by midnight on Friday at the latest. Consider including the following information. Of course, you can say less, or more.

- Your major/minor, reasons for registering, and goals for this course.
- Your math background.
- Any future aspirations involving math. For instance, do you plan to attend graduate school, or pursue a career, in a math-adjacent area?
- Any personal facts you would like to share. For example, I discussed my family, hometown, educational background, hobbies, and pets.
- Any personal circumstances that might impact your performance.
- If you are OK doing so, a recent photo of yourself.
  Pet photos are very welcome!
- Title your message `[math-291] Introduction`. Make certain that your message conforms to the email policies described in our syllabus.

**Thursday** Today, in a guest lecture by Professor Emily Witt, we started class by introducing the notion of a linear equation, which is the sum of constant multiples of variables. After presenting examples and non-examples, we motivated the terminology "linear" by showing that a linear equation in two variables $x, y$ is precisely an equation for a line in the $xy$-plane (and also noticed that a linear equation in three variables defines a plane). Next, we discussed the concept of a system of linear equations, which is a collection of linear equations; by a solution to the system, we mean a solution to every equation. Through examples, we saw that it is possible to have exactly one solution, infinitely many solutions, or no solutions to a given system. In fact, this is the case for every system of linear equations, and we stated this as the Trichotomy Theorem. Next, we discussed elementary operations on systems of linear equations that can be used to simplify the system, with the goal of determining its solutions. Then we introduced the notion of the augmented matrix associated to a system, and defined the so-called "elementary row operations" corresponding to the elementary operations on the system.