Honors Exploration
Diagonalization and applications

Instructions. Show your work, and explain your steps! An unsupported correct answer will earn no credit, yet an incorrect answer supported by some correct logic will receive partial credit. You will be evaluated on the mathematical correctness, and on the quality and clarity of your writing.

Objective. You will discover a useful method for computing arbitrary powers of certain special matrices, the *diagonalizable* matrices, and investigate some interesting applications.

Diagonalization.

(1) Look up the definition of a square *diagonal matrix* on Wikipedia, and summarize it here. Can a square diagonal matrix have an entry of zero somewhere on its diagonal?

(2) Let D be a square $n \times n$ diagonal matrix, with diagonal entries $d_1, d_2, \ldots, d_n$. Compute $\mathsf{D}^2, \mathsf{D}^3$, and discover a formula for D^t for all nonnegative integers t.

We now change topics, but will return to diagonal matrices soon. For the problems below, let A be an $n \times n$ matrix. Furthermore, suppose that $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are eigenvectors of A, and that each eigenvector $\mathbf{p}_i$ is associated to the eigenvalue λ_i. Note: It is possible that the list of eigenvalues $\lambda_1, \ldots, \lambda_n$ may contain repeats. Let P be the matrix whose columns are the list of eigenvectors, and D be the diagonal matrix whose diagonal matrices are the list of eigenvalues, *in the same order*.

(3) What is $\mathsf{A}\mathbf{p}_i$?

(4) Use the result of the previous problem to find a formula for AP.

(5) Directly compute the matrix product PD.

(6) Compare your two answers, and prove the **Theorem**: If P is invertible, then $\mathsf{A} = \mathsf{PDP}^{-1}$.

(7) Look up the definition of a *diagonalizable matrix* on Wikipedia, and compare it to your theorem.

(8) Suppose that P is invertible. Compute $\mathsf{A}^2, \mathsf{A}^3, \mathsf{A}^4$, and deduce a formula for A^k for all nonnegative integers k.

(9) Let's consider a concrete example: Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}.$$

Can you find a formula for A^k where k is any nonnegative integer?

(10) What happens if you try to apply this cool trick to the following matrix?

$$\mathsf{A} = \begin{bmatrix} 4 & 3 \\ 0 & 4 \end{bmatrix}$$

What is the matrix P? What ways can you build the matrix D? What does this example tell you about diagonalizable matrices?

Lines of cats and dogs. Suppose that you have a collection of n treats that you plan to feed to a group of cats and dogs, with each cat (C) receiving one treat, and each dog (D) receiving two treats. These are well-trained pets, and they know to line up when receiving their treats. Set $x_0 = 1$, and for every k at least 1, let x_k be the number of different lines that you can feed *exactly* with k treats. Here, *exactly* refers to the condition that you must not have any leftover treats.

(11) $x_0 = 1$ by definition, and $x_1 = 1$ because the only line that can be fed exactly with one treat is C, but $x_2 = 2$ because of the line CC and the line D. By counting all possible lines of pets, compute the values of x_3, x_4, and x_5.

(12) Derive, *with full justification*, a recursive formula for x_k. To appreciate the recursive nature of this formula, write out the first 10 terms $x_0, \ldots, x_9$.

(13) Express your recursion as a matrix equation involving a 2×2 matrix. In other words, find a matrix F such that $F \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix}$.

(14) Look up the *Golden Ration* in Wikipedia, and write a few words about it. In all future computations, let α be the Golden Ratio, and let β be its conjugate.

(15) Investigate the vectors $F \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ and $F^2 \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ and $F^3 \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$. Can you establish a pattern?

(16) Provide an explicit (i.e., non-recursive) formula for x_k in terms of α and β.

(17) Meditate on your formula for x_k, and verify that it is correct for $x_0, \ldots, x_9$. How does it make you feel? Why would someone perhaps not believe it, at first sight?

(18) How many different lines can you feed with 15 treats?

Stacking chips. You are playing in a low-stakes poker game, and are bored waiting for the other players to complete their turns. Recall that a \$5 dollar chip is red (R), a \$10 chip is blue (B), and a \$25 chip is green (G). To preoccupy yourself, you attempt to answer the following question: *How many different stacks of k chips can you build without consecutive* G's?

To establish notation, define $s_0 = 1$, and for every k at least 1, let s_k count the number of stacks you can build without consecutive G's.

(19) By counting all possible stacks, compute the values of $s_1, \ldots, s_5$.

(20) Derive, *with full justification*, a recursive formula for s_k. To appreciate the recursive nature of this formula, write out the first 10 terms $s_0, \ldots, s_9$.

(21) Express your recursion as a matrix equation involving a 2×2 matrix C.

(22) Describe the pattern you get when you apply powers of C to $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}$.

(23) Provide an explicit (i.e., non-recursive) formula for s_k.

(24) If you have 100 chips, how many stacks can you build without consecutive G's?

Comments. In this worksheet, for simplicity, all of our examples involved only 2×2 matrices, but the principles you have developed here apply to square matrices of arbitrary size. You may gain some practice dealing with 3×3 invertible matrices on Lyryx.