Worksheet: Prime Ideals and Maximality Conditions

Throughout our course, every ring is commutative and contains 1. Throughout this worksheet, R is a ring, and I is an ideal of R.

1. **Warm-up.**
   (a) Recall that the *ideal generated by a subset* $\mathcal{A}$ *of* R is the set of all R-linear combinations of elements in $\mathcal{A}$. Verify that it is indeed an ideal, and is in fact the smallest ideal containing $\mathcal{A}$.
   (b) Given subsets $\mathcal{A}$ and $\mathcal{B}$ of R, let I be the ideal generated by $\mathcal{A}$ and J be the ideal generated by $\mathcal{B}$. Describe, if possible, a generating set for each of the following the ideals in terms of $\mathcal{A}$ and $\mathcal{B}$: $I + J$, IJ, and $I \cap J$.
   (c) A subset $\mathcal{W}$ of R is called *multiplicatively closed* if it contains 1 (the "empty product") and is closed under products. Give examples of multiplicatively closed subsets of an arbitrary ring R.
   (d) Prove that $\mathcal{W} = R \setminus I$ is multiplicatively closed if and only if I is a prime ideal.
   (e) Given $x \in R$, the *colon ideal* with respect to x and I is $(I : x) = \{y \in R \mid xy \in I\}$. Verify that this is an ideal of R, and contains I.

2. Let $\mathcal{W}$ be a multiplicatively closed subset of R, and suppose that I is an ideal maximal with respect to the condition that it is disjoint from $\mathcal{W}$. Prove that I is prime.

   Hint: Suppose otherwise, so that xy is in I for some $x, y \in R$ not contained in I. Use each of x and y to construct an ideal containing I, and then apply the maximality of I.

3. The previous problem establishes a partial converse to one in the Warm-up, provided that there exists an ideal I maximal with respect to being disjoint from $\mathcal{W}$. When does such an ideal exist?

   Hint: Recall Zorn's Lemma.

4. A multiplicatively closed subset $\mathcal{W}$ of a ring R is *saturated* if given $x, y \in R$, $xy \in \mathcal{W} \iff x \in \mathcal{W}$ and $y \in \mathcal{W}$. Prove the following theorem:

   Theorem: Consider a subset $\mathcal{W}$ of a ring R. Then $\mathcal{W}$ is multiplicatively closed and saturated if and only if $R \setminus \mathcal{W}$ is a union of prime ideals of R.

   Hint: For the forward implication, if $\mathcal{W} \subsetneq R$, take $x \in R \setminus \mathcal{W}$ and compute $\langle x \rangle \cap \mathcal{W}$.

5. For a fixed ideal I, consider ideals J of the form $J = (I : x)$ for some $x \in R \setminus I$. Prove that if there exists a maximal element of this form, then it is a prime ideal. **Note**: This result is related to the existence of *associated primes* of an ideal, which we will study later in the semester.

6. Recall that an ideal of R is called *finitely generated* if it is the ideal generated by some finite subset of R. Our goal in this problem is to prove the following:

   Proposition: If I is maximal amongst all proper ideals that are *not* finitely generated, then I is prime.

(a) Assume that I is not prime, and fix $x, y \in R$ such that $xy \in I$ but neither x nor y is in I. Explain why there is a generating set for $\langle x \rangle + I$ of the form $i_1 + r_1 x, \ldots, i_n + r_n x$ for some positive integer n and elements $i_t \in I$.

(b) Explain why the ideal $J = (I : x)$ is finitely generated.

(c) Conclude that I is finitely generated by showing that $I = \langle i_1, \ldots, i_n \rangle + J\langle x \rangle$.

(d) Complete the proof of the proposition.