Worksheet: The Spectrum of a Ring and the Zariski Topology

Throughout this worksheet, R denotes an arbitrary ring; recall that in this course, this means that R is commutative and contains 1.

Definition. The *spectrum* of R, denoted, $\operatorname{Spec} R$, is the set of all prime ideals of R.

Notation. For any subset $\mathcal{A} \subseteq R$, define $\mathbb{V}(\mathcal{A}) = \{P \in \operatorname{Spec} R \mid \mathcal{A} \subseteq P\}$.

1. **Warm-up.**
   (a) Describe $\operatorname{Spec} \mathsf{k}$, where k is a field.
   (b) For arbitrary R, compute $\mathbb{V}(\{0\})$ and $\mathbb{V}(\{1\})$.
   (c) If I is an ideal of R, explain why one can identify $\mathbb{V}(I) \subseteq \operatorname{Spec} R$ with $\operatorname{Spec}(R/I)$.
   (d) Explain why $\operatorname{Spec} R$ has a natural structure of a poset. Describe this poset for $\operatorname{Spec} \mathbb{Z}$.

2. **The Zariski topology.**
   (a) For $\mathcal{A} \subseteq R$, explain why $\mathbb{V}(\mathcal{A}) = \mathbb{V}(I)$, where I is the ideal generated by $\mathcal{A}$. If $I \subseteq J$ are ideals of R, then how does $\mathbb{V}(I)$ compare to $\mathbb{V}(J)$?
   (b) Given ideals I and J of R, write $\mathbb{V}(I) \cap \mathbb{V}(J)$ and $\mathbb{V}(I) \cup \mathbb{V}(J)$ in terms of closed sets of the form $\mathbb{V}(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of R.

   Hint: For the final part, determine inclusions among the set $\mathbb{V}(I) \cup \mathbb{V}(J)$, $\mathbb{V}(I \cap J)$, and $\mathbb{V}(IJ)$. Then show that they must coincide.

   (c) Prove that $\operatorname{Spec} R$ has the structure of a topological space, whose closed sets are those of the form $\mathbb{V}(\mathcal{A})$ for $\mathcal{A} \subseteq R$, which by (a), can also be described as the sets of the form $\mathbb{V}(I)$ for ideals I of R. This is called the *Zariski topology* on $\operatorname{Spec} R$.

   Hint: Recall that the open sets of a topological space must satisfy the following axioms: (1) The empty set is open, (2) Any (possibly infinite) union of open sets is open, and (3) Any finite intersection of open sets is open. Rephrase these axioms in terms of *closed sets*, which, by definition, are complements of open sets.

3. (a) Describe the *closure* of a subset Σ of $\operatorname{Spec} R$, the smallest closed set containing Σ.
   (b) What is the closure of $\{0\}$ in $\operatorname{Spec} \mathbb{Z}$? Describe why your answer means that the prime ideal 0 is a *dense point* of $\operatorname{Spec} \mathbb{Z}$. Then find the closure of $\{P\}$ for all other prime ideals P of $\mathbb{Z}$.
   (c) Sketch the topological space $\operatorname{Spec} \mathbb{Z}$ using your observations from (b). (Be creative!)
   (d) Show that for nonempty $\mathcal{A} \subseteq \mathbb{Z}$, $\mathbb{V}(\mathcal{A}) = \mathbb{V}(\{n\})$, where n is the greatest common divisor among all elements of $\mathcal{A}$.
   (e) Let $\operatorname{maxSpec} R$ denote the subset of $\operatorname{Spec} R$ consisting of maximal ideals. Prove that the subspace topology on $\operatorname{maxSpec} \mathbb{Z}$ is the finite complement topology.

4. Describe the points of $\operatorname{Spec} \mathbb{C}[x]$, and then sketch this spectrum. What is $\operatorname{maxSpec} \mathbb{C}[x]$?

5. Describe, and then sketch, $\operatorname{Spec} \mathbb{R}[x]$.

6. **Map on spectra induced by a ring map.** Let $\phi : R \to S$ be a ring homomorphism.
   (a) Prove that if $P \in \operatorname{Spec} S$, then $\phi^{-1}(P) \in \operatorname{Spec} R$.
   (b) Prove that the function $\phi^{\#} : \operatorname{Spec} S \to \operatorname{Spec} R$ sending P to $\phi^{-1}(P)$ is continuous in the Zariski topology.
   (c) For the ring map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ sending each integer to its residue class modulo 2, describe the induced map on spectra explicitly.
   (d) For the ring map $\mathbb{Z} \hookrightarrow \mathbb{Q}$, describe the induced map on spectra explicitly.

7. Suppose that $P \in \operatorname{Spec} \mathbb{Z}[x]$.
   (a) Describe the possible values of $P \cap \mathbb{Z}$, and describe these possibilities in terms of the induced map $\operatorname{Spec} \mathbb{Z}[x] \to \operatorname{Spec} \mathbb{Z}$. There should be two cases.
   (b) Describe the elements of $\operatorname{Spec} \mathbb{Z}[x]$.
   Hint: Break the problem into the two cases above. In each case, consider the image of P under a natural ring map $\mathbb{Z}[x] \to \mathsf{k}[x]$, where k is a certain field. Recall *Gauss' Lemma.*
   (c) Sketch $\operatorname{Spec} \mathbb{Z}[x]$.

8. **Basic open sets**. For $f \in R$, define $D(f) = \{P \in \operatorname{Spec} R \mid f \notin P\}$.
   (a) Prove that $D(f)$ is an open set of $\operatorname{Spec} R$.
   (b) Prove that the open sets of the form $D(f)$ form a *basis* for the Zariski topology on $\operatorname{Spec} R$–i.e., this is a collection of open sets for which every open set is a union of some elements from this collection.
   (c) For any ideal I of R, prove that $\mathbb{V}(I) = \varnothing$ if and only if $1 \in I$.

9. **Challenge.** Describe $\operatorname{Spec} \mathbb{C}[x, y]$ and $\operatorname{maxSpec} \mathbb{C}[x, y]$. (Feel free to ask me for hints!) Extended Hint: What follows is a rough outline; you may need to modify it, and/or consider isolated special cases. Realize that the action lies in describing the prime ideals P that are nonzero and not principal. For such P, explain why there must exist irreducible polynomials $f, g \in P$ that are relatively prime. Remind yourself of the statement of *Gauss' Lemma*, which you probably already did in solving Problem 7. Apply this to f and g, considered as elements of $\mathbb{C}(x)[y]$, where $\mathbb{C}(x)$ is the the fraction field of $\mathbb{C}[x]$. Remind yourself of the meaning of the term *Bezout's Identity*. Invoke it, and clear denominators, to obtain a nonzero element in $\mathbb{C}[x] \cap P$ and then consider its factors. Conclude by applying symmetry.