# Worksheet: A Module Over a Ring

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Throughout, $R$ is an arbitrary ring (commutative with unity), and $\mathsf{k}$ is a field.

**Definition.** An $R$-*module* $M$ is an abelian group $(M, +)$ with a function $R \times M \to M$, whose image, for $r \in R$ and $u \in M$, is denoted $r \cdot m$, such that for all $r, s \in R$ and $u, v \in M$, the following properties hold.

1. $r \cdot (u + v) = r \cdot u + r \cdot v$.
2. $(r + s) \cdot u = r \cdot u + s \cdot u$.
3. $(rs) \cdot u = r \cdot (s \cdot u)$.
4. $1 \cdot u = u$.

We often refer to the map $R \times M \to M$ as *scalar multiplication* or the *action of* $R$ *on* $M$, and the dot representing this action is often suppressed.

**Definition.** A *submodule* of an $R$-module $M$ is a subgroup $N$ of $M$ under addition that is closed under the action of the ring, i.e., $ru \in N$ for all $r \in R$ and $u \in M$.

1. **Warm-up.**

   (a) Let $M$ be an abelian group, and let $S$ denote the set of all group homomorphisms from $M$ to itself. Check that $S$ is a ring (though it is usually noncommutative). Verify that the existence of a ring homomorphism $R \to S$ is equivalent to the existence of an $R$-module structure on $M$.

   (b) Show that $M = R$ is naturally an $R$-module under the action given by the multiplication of ring elements. What are the $R$ -submodules of $R$?

   (c) We are very familiar with the notion of a $\mathsf{k}$-module. What do we usually call these objects? What about their submodules?

   (d) Let $G$ be any abelian group. Given $x \in G$ and an integer $n$, define an action of $\mathbb{Z}$ on $G$ by $n \cdot x = \underbrace{x + x + \cdots + x}_{n \text{ terms}}$ if $n \geq 0$, and $n \cdot x = \underbrace{-x - x - \cdots - x}_{-n \text{ terms}}$ if $n < 0$. Verify that this makes $G$ a $\mathbb{Z}$-module, and this is the *only* operation $\mathbb{Z} \times G \to G$ for which $G$ an $\mathbb{Z}$-module. What are the submodules of $G$?

2. **Restriction of scalars.** Show that if $\varphi : S \to R$ is a ring homomorphism and $M$ is an $R$-module, then the action of $S$ on $M$ given by $s \cdot u = \varphi(s) \cdot u$ makes $M$ an $S$-module. In particular, if $S$ is a subring of $R$, and $M$ is an $R$-module, then $M$ is also an $S$-module under the same action.

3. Characterize the $\mathsf{k}[x]$-modules, and then use your answer to explain why the same abelian group can have more than one $R$-module structure. Finally, describe the submodules of a given $\mathsf{k}[x]$-module.

   *Hint*: What does restriction of scalars tell us about any $\mathsf{k}[x]$-module? How does a polynomial $p(x)$ over $\mathsf{k}$ act on an element of the module in terms of the action of $x$?

**Definition.** A function $\varphi : M \to N$ between $R$-modules $M$ and $N$ is called an *$R$-module homomorphism* if for all $u, v \in M$ and $r \in R$,

(a) $\varphi(u+v) = \varphi(u) + \varphi(v)$, i.e., $\varphi$ is a group homomorphism, and
(b) $\varphi(ru) = r\varphi(u)$, i.e., $\varphi$ is $R$-linear.

An *$R$-module isomorphism* is an $R$-module homomorphism that is both injective and surjective, and we call two $R$-modules *isomorphic as $R$-modules* if there exists an $R$-module isomorphism between them.

The set of all $R$-module homomorphisms from $M$ to $N$ is denoted $\mathrm{Hom}_R(M, N)$. If $N = M$, $\varphi$ is also called an *endomorphism* of $M$, and we often us $\mathrm{End}_R(M)$ to denote $\mathrm{Hom}_R(M, M)$.

4. What are $\mathsf{k}$-module and $\mathbb{Z}$-module homomorphisms usually called? Then verify that over an arbitrary ring, the kernel $\ker \varphi = \{u \in M \mid \varphi(u) = 0\}$ and the image $\varphi(M)$ are $R$-submodules of $M$ and of $N$, respectively.
5. Identify the natural action of $R$ on $\mathrm{Hom}_R(M, N)$ that makes it an $R$-module.
6. Determine, with justification, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$.

**Definition.** Given $R$-modules $M$ and $N$, the *(external) direct sum* $M \oplus N$ of $M$ and $N$ is the set of coordinate pairs $(u, v)$ for which $u \in M$ and $v \in N$. Addition and scalar multiplication are defined on $M \oplus N$ coordinate-wise, i.e., $(u, v) + (u', v') = (u+u', v+v')$ and $r \cdot (u, v) = (ru, rv)$, for $u, u' \in M$, $v, v' \in N$, and $r \in R$.

More generally, given an arbitrary family of $R$-modules $\{M_i\}_{i \in I}$ indexed by a (possibly infinite) set $I$, the *direct product* $\Pi_{i \in I} M_i$ is the set of coordinate tuples $(u_i)_{i \in I}$ such that $u_i \in M_i$ for each $i \in I$, under coordinate-wise addition and scalar multiplication.

The *(external) direct sum* $\oplus_{i \in I} M_i$ is the subset of this direct product consisting of $(u_i)_{i \in I}$ for which all but finitely many coordinates $u_i = 0$.

7. Check that $M \oplus N$ is an $R$-module, and then do the same for $\oplus_{i \in I} M_i$.
8. Suppose that $M, N$, and $P$ are $R$-modules. Show that as $R$-modules, $\mathrm{Hom}_R(M \oplus N, P) \cong \mathrm{Hom}_R(M, P) \oplus \mathrm{Hom}_R(N, P)$ and $\mathrm{Hom}_R(M, N \oplus P) \cong \mathrm{Hom}_R(M, N) \oplus \mathrm{Hom}_R(M, P)$.
9. Investigate what happens for direct sums and products over arbitrary (possibly infinite) index sets. In particular, describe $\mathrm{Hom}_R(\oplus_I M_i, N)$ and $\mathrm{Hom}_R(M, \Pi_I N_i)$ for arbitrary index sets $I$.

**Definition.** Suppose that $M$ and $N$ are submodules of an $R$-module $P$. We say that $P$ is the *(internal) direct sum* of $M$ and $N$ if every element of $P$ can be written uniquely as the sum of an element of $M$ and an element of $N$.

An $R$-module $P$ is the *(internal) direct sum* of a family of $R$-submodules $\{M_i\}_{i \in I}$ if every element of $P$ can be written uniquely as a sum of elements in the $M_i$, where only finitely many are nonzero.

10. Prove that $M \oplus N$ is the internal direct sum of its submodules $M \oplus 0$ and $0 \oplus N$.