Worksheet: More on Modules

Throughout, R is an arbitrary ring (commutative with unity), M is an R-module, and k is a field.

Definition. We say that M is *generated by* subset $\mathcal{A} \subseteq M$ if every element of M can be expressed as an R-linear combination of elements of $\mathcal{A}$. We call M *finitely generated* if it is generated by a finite set.

Definition. We call M a *free module* if there exists a set $\mathcal{A}$ of M that generates M, and is also R-linearly independent. In this case, we call the set $\mathcal{A}$ a *free basis* for M.

1. **Warm-up.** In each case, verify that M has a natural R-module structure. Then find a minimal generating set for M. Determine if M is finitely generated and/or free, and in the latter case, specify at least two free bases which, if possible, do not differ by units.

   (a) $M = I = \langle a \rangle$, for $a \in R$.

   (b) $M = R[x]$.

   (c) $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ over $R = \mathbb{Z}$.

   (d) $M = \mathsf{k}[x]/\langle x^2+1 \rangle$ over $R = \mathsf{k}$.

   (e) $M = \mathsf{k}[x]/\langle x^2+1 \rangle$ over $R = \mathsf{k}[x]$.

   (f) $M = R^n$, where $R^n = \underbrace{R \oplus \cdots \oplus R}_{n \text{ copies}} \cong \underbrace{R \times \cdots \times R}_{n \text{ copies}}$.

   (g) $M = \bigoplus_{i \in \mathbb{N}} R = R \oplus R \oplus R \oplus \cdots$.

2. Prove that M a free module if and only if it is a direct sum of some number of copies of R, i.e., $M \cong \bigoplus_{i \in \Sigma} R$ for some index set Σ.

Definition. If N is an R-submodule of M, then the abelian group M/N is called the *quotient* of M by N.

3. Verify that M/N inherits a natural R-module structure.

4. Prove that M is finitely generated if and only if it is a quotient module of R^n for some $n \in \mathbb{N}$.

5. Recall that every R-module is an additive abelian group. Recall the isomorphism theorems for groups, and use these to formulate analogous *isomorphism theorems for modules.* Sketch proofs of these theorems.

   Hint: This amounts to verifying that the isomorphims for groups are R-linear.

Definition. An R-*algebra* is a ring S with a ring homomorphism $\alpha : R \to S$. We call α the *structure homomorphism* of the R-algebra S, but we often simply refer to S as an R-algebra without specifying its structure homomorphism.

6. Briefly verify that, with notation as above, S as an R-module under the action given by $r \cdot s = \alpha(r)s$ for all $r \in R$ and $s \in S$.

Definition. A ring homomorphism $\alpha : R \to S$ is called *module finite* if the R-algebra S is a finitely generated R-module via α. In this case, we call S a *module finite R-algebra*, and if α is an injection, then we call α a *module-finite extension of rings*.

7. In each of the following cases, identify a natural R-algebra structure $\alpha : R \to S$ with α injective. Then determine whether S is a module finite extension of R, and whether S is free.
   (a) $R = \mathbb{Q}[x]$ and $S = \mathbb{Q}[x, y, z]$.
   (b) $R = \mathbb{Q}[x, y]$ and $S = \mathbb{Q}[x, y, z]/\langle z^2 - xy\rangle$.
   (c) $R = \mathbb{Q}[x, z]$ and $S = \mathbb{Q}[x, y, z]/\langle z^2 - xy\rangle$.
   (d) $R = \mathbb{Q}[x]$ and $S = \mathbb{Q}[x, y]/\langle xy\rangle$.