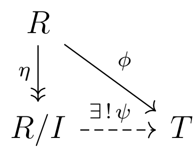
Worksheet: Universal Properties and the Tensor Product

Throughout, R is an arbitrary ring (commutative with unity).

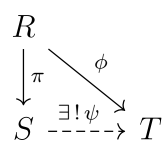
1. **Warm-up: Quotients of rings.** Fix an ideal I of R, and consider the natural surjective ring homomorphism $\eta : R \to R/I$ given by $r \mapsto r + I$.

   (a) Suppose that T is a ring, and $\phi : R \to T$ is a ring homomorphism for which $I \subseteq \ker \phi$. Prove that there exists *unique* (well defined) ring map $\psi : R/I \to T$ for which following diagram commutes, i.e., $\psi \circ \eta = \phi$.

   This is sometimes expressed in words by saying that ϕ *factors uniquely through* η.

   (b) Now suppose that S is a ring and $\pi : R \to S$ is a ring map for which $I \subseteq \ker \pi$. Assume that for any ring map $\phi : R \to T$ for which $I \subseteq \ker \phi$, there exists a *unique* ring map $\psi : S \to T$ making the following diagram commute.

   Prove that S is isomorphic to the ring R/I; in other words, R/I is the *unique* ring (up to isomorphism) satisfying the "universal property" you proved that it satisfies in (a).

   Hint: Your argument should feel pretty general. The uniqueness of ψ is important.

2. **Quotients via universal properties.** Suppose that N is an R-submodule of an R-module M. Call an R-module Q a *candidate quotient* if there exists a map $\pi : M \to Q$ with N contained in the kernel of π, and such that for any other map of R-modules $\phi : M \to L$ with N contained in the kernel of ϕ, there exists a *unique* R-linear map $\psi : Q \to L$ such that $\phi = \psi \circ \pi$.

   (a) Draw a commutative diagram that illustrates this condition.

   (b) Prove that any two candidate quotients must be isomorphic as R-modules.
   Hint: The *uniqueness* part of the universal property is important.

   (c) Verify that the explicitly defined R-module quotient M/N is a candidate quotient.

3. **Direct sums via universal properties.** Consider a sequence $\{M_i\}_{i \in \Sigma}$ of R-modules indexed by some set Σ. We call an R-module D a *candidate direct sum* if there exists a sequence of R-linear maps $\{\alpha_i : M_i \to D\}_{i \in \Sigma}$ such that for any R-module N and sequence of R-linear maps $\{\beta_i : M_i \to N\}$, there exists a *unique* R-linear map $\beta : D \to N$ such that $\beta_i = \beta \circ \alpha_i$ for all $i \in \Sigma$.

   (a) Draw commutative diagram(s) that illustrate this condition.

(b) Prove that any two candidate direct sums must be isomorphic as R-modules.
Hint: Your argument should feel familiar.

(c) Verify that the explicitly defined R-module $\bigoplus_{i\in\Sigma} M_i$ is a candidate direct sum.

(d) What does this say about how to define maps from direct sums? In other words, how are $\{\mathrm{Hom}_R(M_i, N)\}_{i\in\Sigma}$ and $\mathrm{Hom}_R(\bigoplus_{i\in\Sigma} M_i, N)$ related when N is an arbitrary R-module?

4. **Direct products via universal properties.** Consider a set $\{M_i\}_{i\in\Sigma}$ of R-modules.

(a) Formulate what it means for an R-module P to be a *candidate direct product.*
Hint: Think about "projection" maps.

(b) Draw some diagrams that illustrate this condition.

(c) Verify that the explicitly defined R-module $\prod_{i\in\Sigma} M_i$ is a candidate product.

(d) Prove that any two candidate direct products must be isomorphic as R-modules.
Hint: This might feel just a little bit different.

(e) What does this say about how to define maps to products? In other words, when N is an arbitrary R-module, how are $\{\mathrm{Hom}_R(N, M_i)\}_{i\in\Sigma}$ and $\mathrm{Hom}_R(N, \prod_{i\in\Sigma} M_i)$ related?

5. **Tensor products via universal properties.** Consider R-modules M and N. We call T a *candidate tensor product* of M and N if there exists an R-bilinear map $\rho : M \times N \to T$, and every other R-bilinear map $M \times N \to L$ factors *uniquely* through ρ by an R-linear map. Prove that any two candidate tensor products must be isomorphic as R-modules.

6. **Explicit construction of tensor products.** We will now show that a candidate tensor product exists by explicitly constructing one.[1]

(a) Given an arbitrary set Σ, the *free R-module on the set Σ* is just the direct sum $\oplus_{i\in\Sigma} R$. For conceptual purposes, it is often convenient to use different notation to express elements in this direct sum. Indeed, for every element $j \in \Sigma$, use the same symbol j to denote the sequence $(r_i)_{i\in\Sigma}$ defined by $r_i = 0$ if $i \neq j$ and $r_i = 1$ if $i = j$. That is, we use j to denote the sequence in $\oplus_{i\in\Sigma} R$ whose terms are all zero except for the j-th term, which is 1. Convince yourself that with this abuse of notation, we have that

$$(r_i)_{i\in\Sigma} = \sum_{i\in\Sigma} r_i \cdot i.$$

In other words, we may think about the free R-module on the set Σ as all *formal linear combinations* of the elements of Σ with coefficients in R. How do you add such formal linear combinations? How do you multiply one by an element of R? To make things especially concrete, recall that $R[x]$ is the free R-module on the set $\Sigma = \{1, x, x^2, \ldots\}$.

[1]**Note from your instructor.** This construction is pretty involved, but fortunately, almost no one uses it to prove anything. For instance, imagine trying to determine whether a simple tensor $m \otimes n$ is zero. Using the construction, you'd need to check whether (m, n) was in the submodule G, which seems pretty hard. But, what if you could find an R-module L and an R-bilinear map $M \times N \to L$ that sent that particular pair (m, n) to something nonzero? Then, using the universal property, you'd get an R-linear map $M \otimes N \to L$ sending $m \otimes n$ to something nonzero, and hence, $m \otimes n$ must so be nonzero as well! Hopefully, you will get to solve a lot of problems that force you to practice this.

(b) **Sanity check**: Let M, N be two R modules, and consider the set $\Sigma = M \times N$. Observe that the symbols $r \cdot (m,n), (rm, rn), (m,n) + (m',n)$, and $(m+m',n)$ make sense in both F, the free R-module on the set Σ, and also in $M \oplus N$. Compare and contrast these elements in each of these contexts.

(c) We are now prepared to construct the tensor product $M \otimes_R N$ of M and N. Let F be the free R-module on the set $M \times N$, and let G be the submodule of F generated by

$$\begin{aligned}(m+m',n)&-(m,n)-(m',n)\\(m,n+n')&-(m,n)-(m,n')\\r\cdot(m,n)&-(rm,n)\\r\cdot(m,n)&-(m,rn)\end{aligned}$$

where $m, m' \in M, n, n' \in N$, and $r \in R$ are arbitrary. Define $M \otimes_R N$ to be the R-module

$$M \otimes_R N = F/G$$

and for every $m \in M$ and $n \in N$, let $m \otimes_R n$ (we drop the subscript when there is no room for confusion) denote the class of (m,n) in $M \otimes_R N$. We call $m \otimes n$ a simple tensor. What does the fact that all of the terms displayed above say about how to add simple tensors, and multiply them by a ring element? What role do simple tensors play in $M \otimes N$, say in terms of R-module generators?

(d) Verify that $M \otimes N$ is a candidate tensor product, in the sense considered earlier.

Hint: The universal properties for direct sums and quotients may be useful. The point is that to construct a map from $M \otimes N$, you'll need to construct a map on the quotient of a free module.