Worksheet: Working with Tensor Products

Throughout, R is an arbitrary commutative ring with 1.

Universal property of tensor products. Given R-modules M, N, and Q, any *bilinear* (i.e., linear in each factor) map $M \times N \to Q$ of R-modules factors uniquely through the universal bilinear map $M \times N \to M \otimes_R N$ sending $(m, n) \to m \otimes n$.

Basic properties of tensor products. For R-modules M, N, and Q, the following hold.

1. **Identity.** $R \otimes_R M \cong M$ and $M \otimes_R R \cong M$.
2. **Associativity.** $M \otimes_R (N \otimes_R Q) \cong (M \otimes_R N) \otimes_R Q$.
3. **Distributivity.** $M \otimes_R (N \oplus Q) \cong (M \otimes_R N) \oplus (M \otimes_R Q)$.
4. **Commutativity.** $M \otimes_R N \cong N \otimes_R M$.

1. **Warm-up.**
   (a) Prove the basic properties of tensor products.
   (b) Prove that in any tensor product $M \otimes_R N$, any $m \otimes 0 = 0$ and likewise, any $0 \otimes n = 0$. Then prove that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$, and finally, decide whether or not $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ is zero.

Calculations in tensor products. When solving Problems 2–5, you may need to find an appropriate bilinear map. If it is obvious that a map that you define is bilinear, then it is OK to simply assert this. If it isn't obvious (this will be a highly personal decision), or if you are asked in the problem to do so, then please explicitly verify its bilinearity.

2. Show that the element "$2 \otimes 1$" is zero in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.
3. Let I and J be ideals of R.
   (a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.
   (b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ that sends $(r \bmod I) \otimes (r' \bmod J)$ to $rr' \bmod (I + J)$.
4. Let $I = \langle 2, x \rangle$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. The ring $R/I \cong \mathbb{Z}/2\mathbb{Z}$ is naturally an R-module *annihilated by* both 2 and x, meaning that the R-module action given by these elements is zero.
   (a) Show that the map $\varphi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined by
   $$\varphi(a_0 + a_1x + \cdots + a_nx^n, b_0 + b_1x + \cdots + b_mx^m) = a_0/2 \cdot b_1 \bmod 2$$
   is R-bilinear.
   (b) Show that there is an R-module homomorphism from $I \otimes_R I$ to $\mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $p(0)/2 \cdot q'(0)$, where q' denotes the usual polynomial derivative of q.
   (c) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

(d) Show that $2 \otimes 2 + x \otimes x$ cannot be written as a simple tensor in $I \otimes_R I$.

Hint: You may want to use a new bilinear map!

5. **Tensor products of vector spaces.** Let V and W be finite dimensional vector spaces over a field k of dimensions m and n, respectively.

   (a) Fix bases so that $V \cong \mathsf{k}^m$ and $W \cong \mathsf{k}^n$ are identified with spaces of column vectors in the usual way. Using the matrix multiplication map
$$\begin{aligned} \mathsf{k}^m \times \mathsf{k}^n &\to \mathsf{k}^{mn} \\ (\mathbf{v},\mathbf{w}) &\mapsto \mathbf{v} \cdot \mathbf{w}^{\mathrm{tr}} \end{aligned}$$
   prove that $\mathsf{k}^m \otimes_{\mathsf{k}} \mathsf{k}^n$ is naturally[1] isomorphic to the space $\mathsf{k}^{m\times n}$ of $m \times n$ matrices.

   (b) Conclude that $V \otimes_{\mathsf{k}} W$ has dimension mn, and describe an explicit basis in terms of bases for V and W.

   (c) Let $X \subseteq \mathsf{k}^{m\times n}$ be the image of the bilinear map described in (a). Explain why X consists of the matrices of rank at most 1.

   (d) How likely is it that a randomly chosen element of $V \otimes_{\mathsf{k}} W$ can be written as some $\mathbf{v} \otimes \mathbf{w}$?

   Hint: Use (c) to “measure” the size of the set of simple tensors within the tensor product.

   (e) Let $\{e_1, e_2\}$ be a basis for $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ of $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

6. **Tensor products of free modules.** The *rank* of a free module is the size of its free basis. Explain why the tensor product of finitely generated modules is finitely generated, and why the tensor product of free modules (of rank m and n respectively) is free (of rank mn).

7. **Adjointness of tensor and Hom** Let $R \to S$ be a ring homomorphism. Let M and N be S modules, and let Q be an R-module.

   (a) Discuss natural R-module structures on M and N.

   (b) Discuss a natural S-module structure on $\mathrm{Hom}_R(N, Q)$.

   (c) Given a R-bilinear map $M \times N \to Q$, describe a natural R-module map $M \to \mathrm{Hom}_R(N, Q)$.

   (d) Show that there is an R-module isomorphism
$$\mathrm{Hom}_R(M \otimes_R N, Q) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, Q)).$$

   (e) Show that there is an S-module isomorphism
$$\mathrm{Hom}_R(M \otimes_S N, Q) \cong \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, Q)).$$

8. **Tensor products of algebras.** Let A and B be R-algebras.

   (a) Show that $A \otimes_R B$ has the structure of an R-algebra.

[1] meaning that the isomorphism can be described without choosing a basis.

(b) An R-*algebra homomorphism* between R-algebras is a ring homomorphism that is also an R-module homomorphism. Show that there exist R-algebra homomorphisms $A \to A \otimes_R B$ and $B \to A \otimes_R B$ that make $A \otimes_R B$ into a *coproduct* in the category of R-algebras, meaning that given any R-algebra C to which both A and B map, there exists a unique R-algebra map $A \otimes_R B \to C$ making the relevant diagrams commute.

(c) What is the coproduct in the category of sets? What about the category of modules?