Worksheet: Localization of Rings

Throughout, R is an arbitrary commutative ring with unity. Recall that a subset $\mathcal{W}$ of R is *multiplicatively closed* if it is closed under products and contains 1.

Definition. Given a multiplicatively closed subset $\mathcal{W}$ of R, the *localization* of R at $\mathcal{W}$ is a ring[1], denoted $\mathcal{W}^{-1}R$, having the following universal property: For all ring homomorphisms $\varphi : R \to S$ such that $\varphi(\mathcal{W})$ consists of units in S, ϕ factors uniquely, as a ring map, through $R \to \mathcal{W}^{-1}R$. The ring $\mathcal{W}^{-1}R$ and the homomorphism $R \to \mathcal{W}^{-1}R$ are unique up to unique isomorphism.

Concretely: The localization $\mathcal{W}^{-1}R$ is the set of equivalence classes $\frac{r}{w}$ of pairs $(r, w) \in R \times \mathcal{W}$ under the following equivalence relation:

$$(r_1, w_1) \sim (r_2, w_2) \text{ if there exists } w \in \mathcal{W} \text{ such that } w(r_1 w_2 - r_2 w_1) = 0.$$

1. **Warm-up.**
   (a) Describe what it means for an algebraic structure to be unique up to unique isomorphism.
   (b) What is the ring structure on $\mathcal{W}^{-1}R$ and why is it *well defined*? Describe the maps in the above factorization explicitly and understand why they are uniquely determined.
   (c) Discuss good ways to understand $\mathcal{W}^{-1}R$ explicitly in the following cases:
      i. $R = \mathbb{Z}$ and $\mathcal{W} = \mathbb{Z} \setminus \{0\}$.
      ii. R is *any* domain and $\mathcal{W} = R \setminus \{0\}$.
      iii. $R = \mathbb{Z}$ and $\mathcal{W} = \{2^n \mid n \geq 0\}$.
      iv. $R = \mathbb{Z}$ and $\mathcal{W} = \mathbb{Z} \setminus 2\mathbb{Z}$.
      v. $R = \mathbb{Z}/12\mathbb{Z}$ and $\mathcal{W} = \{\overline{1}, \overline{2}, \overline{4}, \overline{8}\}$.
      vi. For a fixed integer $n \neq 0$, $R = \mathbb{Z}[x]/\langle nx - 1\rangle$ and $\mathcal{W} = \{n^k \mid k \geq 0\}$.
   (d) Explain how to think of $\mathcal{W}^{-1}R$ as a quotient of the polynomial ring $R[\{X_w\}_{w \in \mathcal{W}}]$.

2. **The kernel of localization.**
   (a) Characterize when localization $R \to \mathcal{W}^{-1}R$ is injective in terms of $\mathcal{W}$.
   (b) If $\mathcal{W} \subseteq R$ is multiplicatively closed, define $I = \{r \in R \mid rw = 0 \text{ for some } w \in \mathcal{W}\}$. Prove that the localization map $R \to \mathcal{W}^{-1}R$ factors through the quotient map $R \to R/I$.
   (c) Examine what this says about the example in Problem 1(b)v.

Definition. Let $\varphi : R \to S$ be a ring homomorphism. The *contraction* (to R) of an ideal J of S, denoted $J \cap R$, is the ideal $\varphi^{-1}(J)$ of R. The *expansion* (to S) of an ideal I of R, denoted IS, is the ideal of S generated by $\varphi(I)$.

3. **Contraction and expansion.**
   (a) Show that $(J \cap R)S \subseteq J$ for all ideals J of S.
   (b) Show that $I \subseteq (IS) \cap R$ for all ideals I of R.
   (c) Give an example of a ring map $R \to S$ and a prime P of R such that PS is *not* prime.

[1]Or more precisely, a *ring homomorphism* $R \to \mathcal{W}^{-1}R$

(d) Given $P \in \operatorname{Spec} R$ and $\mathcal{W} \subseteq R$ multiplicatively closed, suppose that $P \cap \mathcal{W} = \varnothing$. Prove that the expansion $P(\mathcal{W}^{-1}R)$ is prime. What happens if $P \cap \mathcal{W} \neq \varnothing$?

Definition. A ring is *local* if it has a unique maximal ideal.

4. **The spectrum of a localization.** Let $\mathcal{W} \subseteq R$ be multiplicatively closed.
   (a) Show that $\operatorname{Spec} \mathcal{W}^{-1}R$ is homeomorphic to $\{P \in \operatorname{Spec} R \mid P \cap \mathcal{W} = \varnothing\}$ with the subspace topology.

   Hint: The maps are given by contraction and expansion.

   (b) Given $f \in R$, R_f denotes the localization of R at the multiplicatively closed set of positive powers of f. Prove that $\operatorname{Spec} R_f$ is homeomorphic to the complement, $D(f)$, of $\mathbb{V}(f)$ in $\operatorname{Spec} R$.

   (c) For $P \in \operatorname{Spec} R$, recall that $\mathcal{W} = R \setminus P$ is multiplicatively closed. Prove that in this case, $\mathcal{W}^{-1}R$ is a local ring. We call this ring the *localization of* R *at* P, and denote it R_P.

5. **Fields determined by prime ideals.** Given $P \in \operatorname{Spec} R$, there are two natural ways to construct a field. The first is to take the fraction field of the domain R/P. The second way is to take the quotient of the local ring R_P by its unique maximal ideal, which we denote $k(P)$. Prove that these two natural fields constructed from P are isomorphic by constructing an explicit isomorphism between them. Make certain to verify that your maps are well-defined!

Definition. Recall that the *radical* of an ideal I of R is $\sqrt{I} = \{f \in R \mid f^n \in I \text{ for some } n \geq 1\}$. When $I = 0$ is the zero ideal, we call $\sqrt{I} = \sqrt{0}$ the *nilradical* of R, and sometimes denote it $\operatorname{Nil}(R)$.

Definition. Recall that a nonzero element of a ring is *nilpotent* if some positive power of it is zero. A ring is called *reduced* if it contains no nilpotent elements.

6. **Radicals, nilradicals, and nilpotents in terms of prime ideals.**
   (a) Prove that $f \in R$ is nilpotent if and only if f is contained in *every* prime ideal of R.

   Hint: If f is not nilpotent, then what does this say about R_f? You might want to call upon the fact that every nonzero ring contains a maximal ideal.

   (b) Conclude that
   $$\operatorname{Nil}(R) = \bigcap_{P \in \operatorname{Spec} R} P = \bigcap_{P \in \operatorname{minSpec} R} P$$
   where $\operatorname{minSpec} R \subseteq \operatorname{Spec} R$ is the set of primes that are minimal with respect to inclusion.

   (c) Conclude that for any ideal I of R, $\sqrt{I} = \bigcap_{P \in \mathbb{V}(I)} P$.

   Hint: What is the expansion of $\sqrt{I}$ in R/I?