Worksheet: Exactness

Throughout, R is an arbitrary commutative ring with unity, and M, N, P and Q are R-modules.

Definition. A pair of R-module homomorphisms

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

is called *exact* if the image of φ coincides with the kernel of ψ. An *exact sequence* is a sequence of R-module homomorphisms

$$\cdots \to M_{i-1} \xrightarrow{\varphi_i} M_i \xrightarrow{\varphi_{i+1}} M_{i+1} \to \cdots$$

(either finite or infinite) , for which every pair $M_{i-1} \xrightarrow{\varphi_i} M_i \xrightarrow{\varphi_{i+1}} M_{i+1}$ is exact. A *short exact sequence* is an exact sequence of the form

$$0 \to M \xrightarrow{\varphi} N \xrightarrow{\psi} P \to 0$$

and the term *long exact sequence* is often used for an arbitrary exact sequence.

1. **Warm-up.**
   (a) Suppose that M is isomorphic to a submodule N' of N, and P is isomorphic to the quotient N/N'. Prove that there exists an exact sequence of the form $M \to N \to P$.
   (b) Characterize the existence of exact sequences of each of the following form:
      i. $0 \to M \to N$.
      ii. $N \to P \to 0$.
      iii. $0 \to M \to N \to 0$.
      iv. A short exact sequence $0 \to M \to N \to P \to 0$.
   (c) Construct a natural short exact sequence with the following terms, in some order:
      i. R, I and R/I, if I is an ideal of R.
      ii. $M, \ker \varphi$, and $\varphi(M)$, if $\varphi : M \to N$ is an R-module map.
      iii. M, N, and $M \oplus N$.
      iv. $I + J, I \cap J$, and $I \oplus J$, where I and J are ideals of R.
      v. $R/(I + J), R/(I \cap J)$, and $R/I \oplus R/J$, where I and J are ideals of R.
   (d) For which $x \in R$ is the following exact, where the map $R \to R/\langle x \rangle$ is the natural quotient homomorphism?
   $$0 \to R \xrightarrow{\cdot x} R \to R/\langle x \rangle \to 0$$

2. **Left exactness of the** Hom **functor.**
   (a) Given an R-module map $\varphi : M \to N$, find naturally-induced maps $\operatorname{Hom}_R(Q, M) \to \operatorname{Hom}_R(Q, N)$ and $\operatorname{Hom}_R(N, Q) \to \operatorname{Hom}_R(M, Q)$.
   (b) Prove that if $0 \to M \to N \to P$ is exact, then the induced sequence
   $$0 \to \operatorname{Hom}_R(Q, M) \to \operatorname{Hom}_R(Q, N) \to \operatorname{Hom}_R(Q, P)$$
   is also exact.

(c) Prove that if $M \to N \to P \to 0$ is exact, then the induced sequence

$$0 \to \operatorname{Hom}_R(P, Q) \to \operatorname{Hom}_R(N, Q) \to \operatorname{Hom}_R(M, Q)$$

is also exact.

(d) Find examples that show that, if you modify the Hom sequences appearing (b) and (c) by adding a zero at the end, the resulting sequences need not be short exact.

Definition. A *complex* of R-modules is a sequence of R-module homomorphisms

$$\cdots \to M_{i-1} \xrightarrow{\varphi_i} M_i \xrightarrow{\varphi_{i+1}} M_{i+1} \to \cdots$$

for which each $\operatorname{Im}(\varphi_i) \subseteq \ker(\varphi_{i+1})$.

3. **Some basic facts.**

(a) Given maps $\varphi : M \to N$ and $\psi : M' \to N'$ of R-modules, construct the tensor product of φ and ψ, that is, an R-module map $\varphi \otimes \psi : M \otimes_R M' \to N \otimes_R N'$ with $m \otimes n \mapsto \varphi(m) \otimes \psi(n)$.

(b) Make the following statement precise: One may take the tensor product of a complex of R-modules with an arbitrary R-module M to obtain another complex of R-modules. Justify that what you get is actually a complex.

Hint: Tensor each of the maps in the complex with the identity $\operatorname{id} : M \to M$.

(c) What complex does one get after tensoring the sequence from (d) in the warm-up with an arbitrary R-module?

Note: To give a precise answer, you'll need to formulate what it means for two complexes (or for this problem, complexes with at most three nonzero terms) of R-modules to be isomorphic.

4. **Exactness properties of tensor product.** Consider an arbitrary R-module Q, and a short exact sequence of R-modules $0 \to M \xrightarrow{\varphi} N \xrightarrow{\psi} P \to 0$. Consider the new complex

$$0 \to M \otimes_R Q \xrightarrow{\varphi \otimes \operatorname{id}_Q} N \otimes_R Q \xrightarrow{\psi \otimes \operatorname{id}_Q} P \otimes_R Q \to 0.$$

(a) Prove that $\psi \otimes \operatorname{id}_Q$ is surjective.

(b) Based on your earlier work, the new complex is, in fact, a complex. Hence $\operatorname{Im}(\varphi \otimes \operatorname{id}_Q) \subseteq \ker(\psi \otimes \operatorname{id}_R)$. Prove that these two submodules of $N \otimes_R Q$ are equal, that is, prove that the new complex is exact at $N \otimes_R Q$.

Hint: Use the exactness of the original complex to construct a canonical map $P \otimes_R Q \to N \otimes Q/(\operatorname{Im}(\varphi \otimes \operatorname{id}_Q))$. Make sure to check that the map is well-defined! Next, consider the composition $N \otimes_R Q \xrightarrow{\psi \otimes \operatorname{id}_Q} P \otimes_R Q \to N \otimes_R Q/(\operatorname{Im}(\varphi \otimes \operatorname{id}_Q))$. Where does this composition send an element $\sum_{\text{finite}} n_i \otimes q_i \in \ker(\psi \otimes \operatorname{id}_Q)$?

(c) Construct an example to illustrate that the new complex need not be exact at $M \otimes_R Q$.

Hint: The last part of the Problem 3 may be useful.