Worksheet: Localization of Modules

Throughout, R and S are arbitrary commutative rings with unity, $\mathcal{W}$ is a multiplicatively closed subset of R, and M, N, and Q are R-modules unless otherwise specified.

Definition. The *localization* of M at $\mathcal{W}$ is an R-module $\mathcal{W}$, denoted $\mathcal{W}^{-1}M$, is the set of equivalence classes $\frac{u}{w}$ of pairs $(u, w) \in M \times \mathcal{W}$ under the following equivalence relation:

$$(u_1, w_1) \sim (u_2, w_2) \text{ if there exists } w \in \mathcal{W} \text{ such that } w \cdot (w_2 \cdot u_1 - w_1 \cdot u_2) = 0.$$

1. **Warm-up.**
   (a) Verify that $\mathcal{W}^{-1}M$ is an R-module, and that there is a natural R-module homomorphism $M \to \mathcal{W}^{-1}M$.
   (b) Find a universal property for the localization of M at $\mathcal{W}$.
   Hint: Your universal property might involve the bijectivity of certain multiplication maps. To get the idea, it might help to recall the universal property of $\mathcal{W}^{-1}R$.
   (c) Find a natural action under which $\mathcal{W}^{-1}M$ is a $\mathcal{W}^{-1}R$-module, and make sure it is well defined.

2. **Extension of scalars.** Given a ring homomorphism $R \to S$, recall that S is an R-module via restriction of scalars. Prove that for any R-module M, the R-module $S \otimes_R M$ is naturally an S-module.

3. **Localization as the extension of scalars.** Write a precise mathematical formula for the following statement, and prove it:

   $\mathcal{W}^{-1}M$ is the $\mathcal{W}^{-1}R$-module obtained by extension of scalars of an R-module M.

4. **Exactness of localization.**
   (a) Given an R-module homomorphism $\varphi : M \to N$, find a naturally-induced $\mathcal{W}^{-1}R$-module homomorphism $\mathcal{W}^{-1}\varphi : \mathcal{W}^{-1}M \to \mathcal{W}^{-1}N$.
   (b) Explain how the induced map from (a) relates to the following map:
   $$\mathrm{id}_{\mathcal{W}^{-1}R} \otimes \varphi : \mathcal{W}^{-1}R \otimes_R M \to \mathcal{W}^{-1}R \otimes_R N.$$
   (c) Prove that localization of modules is *exact* in the sense that given any short exact sequence $0 \to M \to N \to Q \to 0$, the following induced sequence is also exact:
   $$0 \to \mathcal{W}^{-1}M \to \mathcal{W}^{-1}N \to \mathcal{W}^{-1}Q \to 0.$$
   (d) Conclude that if I is an ideal of R, then $\mathcal{W}^{-1}(R/I)$ and $(\mathcal{W}^{-1}R)/(\mathcal{W}^{-1}I)$ are isomorphic $\mathcal{W}^{-1}R$-modules.

5. **Local properties.** If P is a prime ideal of R, we use M_P to denote $(R \setminus P)^{-1}M$.

(a) Given any R-module map $\varphi : M \to N$, its *cokernel* is the quotient $\operatorname{coker}\varphi := N/\operatorname{Im}\varphi$. Prove that the formation of the kernel, cokernel, and image of φ commute with localization for all $P \in \operatorname{Spec} R$, i.e., the following hold, where $(-)_P$ denotes $(R \setminus P)^{-1}(-)$:

$$(\ker\varphi)_P = \ker(\varphi_P), (\operatorname{coker}\varphi)_P = \operatorname{coker}(\varphi_P), (\operatorname{Im}\varphi)_P = \operatorname{Im}(\varphi_P).$$

(b) For any $u \in M$, its *annihilator* is $\operatorname{ann}_R u = \{r \in R \mid r \cdot u = 0\}$. Verify that $\operatorname{ann}_R u$ is an ideal of R, and is proper if and only if $u \neq 0$.

(c) Show that $\frac{u}{1} \in M_P$ is nonzero if and only if $\operatorname{ann}_R u \subseteq P$.

(d) Show that $u = 0$ if and only if $\frac{u}{1} \in M_P$ is zero for all $P \in \operatorname{Spec} R$.

(e) Prove the same statement as in (d) after replacing $\operatorname{Spec} R$ with $\operatorname{maxSpec} R$.

(f) For any R-module map $\varphi : M \to N$, prove that φ is injective (respectively, surjective) if and only if φ_P is injective (respectively, surjective) for every $P \in \operatorname{Spec} R$ (equivalently, every $P \in \operatorname{maxSpec} R$).

(g) Prove that submodules Q, Q' of N satisfy $Q \subseteq Q'$ if and only if $Q_P \subseteq (Q')_P$ for all $P \in \operatorname{Spec} R$ (equivalently, every $P \in \operatorname{maxSpec} R$).

Hint: $Q \subseteq Q'$ if and only if $(Q + Q')/Q' = 0$.

(h) Prove the following theorem:

Theorem. *A sequence of R-modules $0 \to M \to N \to Q \to 0$ is exact if and only if*

$$0 \to M_P \to N_P \to Q_P \to 0$$

is exact for all $P \in \operatorname{Spec} R$ (equivalently, all $P \in \operatorname{maxSpec} R$).

6. **Localization commutes with tensor product.** Prove, or find a counterexample, to the following statement: $\mathcal{W}^{-1}(M \otimes_R N) \cong \mathcal{W}^{-1}M \otimes_{\mathcal{W}^{-1}R} \mathcal{W}^{-1}N$.

   Hint: You may want to start by using universal properties.