Worksheet: Flatness

Throughout, R is a commutative ring with unity.

Definition. We call an R-module *flat* if for any short exact sequence $0 \to N \to P \to Q \to 0$, the induced sequence

$$0 \to N \otimes_R M \to P \otimes_R M \to Q \otimes_R M \to 0$$

is also exact.

1. **Warm-up.**
   (a) Characterize flatness of M in terms of $\varphi \otimes \mathrm{id}_M$, for injection of R-modules φ.
   Hint: You might want to call upon a problem from the worksheet on exactness.
   (b) Given a multiplicatively closed set $\mathcal{W}$ of R, when is $\mathcal{W}^{-1}R$ flat?
   Hint: You might want to call upon a problem from the worksheet on localization of modules.
   (c) Prove that all free modules are flat.
   (d) Show that $\mathbb{Z}/2\mathbb{Z}$ is not flat over $\mathbb{Z}$. *Hint*: Consider the inclusion $2\mathbb{Z} \subseteq \mathbb{Z}$.

Theorem. *An R-module M is flat if and only if for any exact sequence of R-modules*

$$\cdots \to N_{i-1} \xrightarrow{\varphi_{i-1}} N_i \xrightarrow{\varphi_i} N_{i+1} \to \cdots$$

of arbitrary length, the induced sequence

$$\cdots \to N_{i-1} \otimes_R M \xrightarrow{\varphi_{i-1} \otimes \mathrm{id}_M} N_i \otimes_R M \xrightarrow{\varphi_i \otimes \mathrm{id}_M} N_{i+1} \otimes_R M \to \cdots$$

is also exact. Note: The original indices on the φ maps were (consistently) incorrect. This has been corrected.

2. **Tensoring with flat modules preserve any exact sequence.**
   (a) Following the notation in the theorem above, the inclusion $\operatorname{Im} \varphi_i \subseteq N_{i+1}$ induces a map $(\operatorname{Im} \varphi_i) \otimes_R M \to N_{i+1} \otimes_R M$ given on simple tensors by $w \otimes u \mapsto w \otimes u$. Prove that the image of this induced map is $\operatorname{Im}(\varphi_i \otimes \mathrm{id}_M)$, and explain why the induced map is an isomorphism onto this image.
   (b) The inclusion $\ker \varphi_i \subseteq N_i$ induces a map $(\ker \varphi_i) \otimes_R M \to N_i \otimes_R M$ given on simple tensors by $w \otimes u \mapsto w \otimes u$. Prove that the image of this induced map is $\ker(\varphi_i \otimes \mathrm{id}_M)$, and explain why the induced map is an isomorphism onto this image.
   Hint: Consider the short exact sequence $0 \to \ker \varphi_i \to N_i \to \operatorname{Im} \varphi_i \to 0$. Tensor this with M over R, and use the (a) to obtain a complex that is isomorphic, as complexes, to $0 \to (\ker \varphi_i) \otimes M \to N_i \otimes M \to \operatorname{Im}(\varphi_i \otimes \mathrm{id}_M) \to 0$.
   (c) Use (a), (b), and the exactness of the original sequence to prove the non-trivial direction of the above theorem.

3. **Useful properties of flatness.**

(a) Given an ideal I of R and an R-module M, let IM denote the the subset of M consisting of all R-linear combinations of elements of M with coefficients in I. Verify that this is a submodule of M, and prove that, if M is flat, then $(I \cap J)M = IM \cap JM$.

Hint: Start with a short exact sequence from the worksheet on exact sequences.

(b) Fix $x \in R$ and an ideal I of R. Prove that if S is a *flat* R-algebra (i.e., it is an R-algebra that is a flat R-module), then the ideals $(I :_R x)S$ and $(IS :_S x)$ (each subscript specifies the ring in which we are taking the colon) of S are equal.

Hint: Realize $(I :_R x)$ as the kernel of a map $R \to R/I$. Is your map necessarily surjective? If not, try using the sequence $0 \to (I :_R x) \to R \to R/I$. (Yes, the omission of the final zero is intentional!)

(c) We will later prove, under the condition that R is *Noetherian*, that if S is a flat R-algebra, M, N are R-modules, and if M is finitely generated, then Hom commutes with flat base change. That is, the natural homomorphism

$$S \otimes_R \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)$$

is an isomorphism. Specify this natural homomorphism.