Worksheet: Dimension and Integral Extensions

Throughout, R and S are commutative rings with unity.

Definition. Given $P_i \in \operatorname{Spec}(R)$, $1 \leq i \leq n$, for which

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n,$$

we call n the *length* of this chain of prime ideals.

Definition. The *Krull dimension of* R, which is often referred simply the *dimension of* R and denoted $\dim R$, is the supremum of the lengths of all chains of prime ideals of R.

1. **Warm-up.**
   (a) Find the Krull dimension of each of the following rings:
      i. $\mathbb{Z}$.
      ii. $\mathbb{Q}[x]$.
      iii. The field $\mathbb{C}(t)$ of meromorphic functions on the Riemann sphere.
   (b) What is the dimension of any field? What about a PID that is not a field? What can be said about the dimension of a domain that is not a field?
   (c) Guess the dimension of $\mathsf{k}[x_1, \ldots, x_n]$, and prove that your guess is a lower bound for the actual dimension. (It is not easy to show that it is an upper bound, but we will do this soon.)
   (d) Find a ring R with infinite Krull dimension, justifying the fact that "supremum" in the definition of Krull dimension cannot be replaced with "maximum."

2. **The dimension of a products of rings.** Determine $\dim(R \times S)$ in terms of $\dim R$ and $\dim S$. Then compare your answer to the formula for the dimension of a product of vector spaces.

3. **Dimensions of quotients and localizations.** Suppose that I is an ideal of R, and $\mathcal{W}$ is a multiplicative subset of R.
   (a) Explain why $\dim(R/I)$ and $\dim(\mathcal{W}^{-1}R)$ are each at most $\dim R$.
   (b) Suppose that R is finite dimensional. Are all values less than $\dim R$ achieved as the dimension of some quotient of R?
   (c) Suppose that R is finite dimensional. Are all values less than $\dim R$ achieved as the dimension of some localization of R?
   (d) Construct a zero-dimensional ring that is the quotient of $\mathsf{k}[x]$ modulo a non-maximal ideal. Then do the same for $\mathsf{k}[x, y]$. Can you generalize this to any number of variables?

Theorem (Going up). *Suppose that $R \subseteq S$ is integral. Then for any chain of prime ideals*

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

of R, there exists a chain of prime ideals

$$Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$$

of S such that Q_i lies over P_i for each $1 \leq i \leq n$.

4. Prove the Going up theorem.

   Hint: Apply induction on n. To get the idea for the inductive step, you might first try proving the $n = 1$ case from the base case. The ideas from the proof of Part 3 of the Lying over theorem could be helpful here.

Corollary 1 (Integral extensions preserve dimension)**.** *If* $R \subseteq S$ *is integral, then* $\dim R = \dim S$.

5. Prove the above corollary.

   Hint: The Going up theorem will give you an inequality between $\dim R$ and $\dim S$. To get the other, look back at the statement of the Lying over theorem.