Worksheet: Normality and Direct Summands

Throughout, R and S are commutative rings with unity, and k denotes a field.

1. **Warm-up.** The following aim to help solidify some recently-investigated concepts.
   (a) Assuming your conjectured value of the dimension of a polynomial ring over a field, compute the dimension of the ring
   $$\mathbb{Q}[x,y,z,w]/\langle z^3 - x^3z^2 + z - y^5, w^5 - wx^4 - w^2y^3 - w^3z^2 - 5\rangle.$$
   Hint: Can you realize this as the composition of two integral extensions?
   (b) Prove that if $\mathcal{W}$ is a multiplicatively closed subset of R that contains at least one nonzero non-unit, then the localization map $R \to \mathcal{W}^{-1}R$ can never be integral.
   Hint: Argue by contradiction, and look at the induced map $\operatorname{Spec}(U^{-1}R) \to \operatorname{Spec} R$.

Definition. A domain R is *normal* if the only elements of its fraction field that are integral over R are the elements of R itself.

Proposition. *Every UFD is normal.*

2. Prove the proposition. *Hint*: Write an integral element of the fraction field in lowest terms.

3. Which of the following are normal?
   (a) $\mathbb{Z}$.
   (b) $\mathbb{Z}[x,y,z]$.
   (c) $\mathbb{Z}[\sqrt{5}]$. *Hint*: Consider the element $\frac{1+\sqrt{5}}{2}$.
   (d) The algebraic integers A, i.e., the subring of $\mathbb{C}$ of numbers that are integral over $\mathbb{Z}$.
   Hint: Remember–again–that the composition of integral extensions is again integral.
   (e) $\mathsf{k}[x^2, x^3]$, the largest subring of $\mathsf{k}[x]$ containing k and the monomials x^2, x^3. In other words, this ring consists of all polynomial expressions in x^2 and x^3 with coefficients in k, so that, e.g., $x^7 + x^6 = (x^2)^2x^3 + (x^3)^2 \in \mathsf{k}[x^2, x^3]$.

Definition. Given an extension of rings $R \subseteq S$ making S an R-module, we say that R is a *direct summand* of S if there is an R-linear map $\pi : S \to R$ with $\pi(1) = 1$. In other words, the map π satisfies the following conditions: $\pi(s + s') = \pi(s) + \pi(s')$ and $\pi(rs) = r\pi(s)$ for every $s, s' \in S$ and $r \in R$, and $\pi(1) = 1$.

4. **Direct summands.** Adopt the notation from the above definition.
   (a) Show that the restriction of π to R is the identity map on R.
   (b) Prove that every element $s \in S$ can be written *uniquely* as $s = r + t$ with $r \in R$ and $t \in \ker \pi$.
   Hint: There are two parts to this: existence of such s, t, and their uniqueness. For the first part, rewrite zero in a creative way in terms of π and s.
   (c) Conclude that if R is a direct summand of S, then S is isomorphic to $R \oplus \ker \pi$. Note that this explains the use of the terminology *direct summand* in this context.

(d) Prove that R is a direct summand of S if and only if there exists an R-module M and an R-module isomorphism $S \cong R \oplus M$.

(e) Prove that $R = \mathsf{k}[x, y]$ is a direct summand of $S = \mathsf{k}[x, y, z, w]$. How does this generalize?

(f) We will now show that $R = \mathsf{k}[x^2, xy, y^2]$ is a direct summand of $S = \mathsf{k}[x, y]$.

i. Prove that $R = \operatorname{span}_{\mathsf{k}}\{x^i y^j : i + j \text{ is even}\}$.

ii. Verify that $M = \operatorname{span}_{\mathsf{k}}\{x^i y^j : i + j \text{ is odd}\}$ is not closed under multiplication, but is closed under multiplication by elements of R. Conclude that M is an R-submodule of S.

iii. Conclude that R is a direct summand of S.

(g) Test your intuition, and generalize the above. Can we can replace S with a polynomial ring in any number of variables? What should R be? Is it necessary to only look at monomials of degree 2? You don't have to to prove your conjecture, but you can if you'd like.

5. **Direct summands and normality.**

(a) Consider an inclusion of domains $R \subseteq S$. Prove that if S is normal and R is a direct summand of S, then R must also be normal.

(b) Conclude that $R = \mathsf{k}[x^2, xy, y^2]$ is normal.

(c) What generalization of this does your conjecture from the previous problem suggest?

6. **A useful fact.** Suppose that R is a normal domain, $K = \operatorname{frac} R$ is its fraction field, and L is the algebraic closure of k, so that

$$R \subseteq K \subseteq L.$$

Simple examples of this situation are $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \overline{\mathbb{Q}}$, or $\mathsf{k}[x] \subseteq \mathsf{k}(x) \subseteq \overline{\mathsf{k}(x)}$, where in both cases, the overbar denotes algebraic closure. Throughout this problem, fix an element $\ell \in L$.

(a) By definition, the element $\ell \in L$ must satisfy a monic polynomial with coefficients in K, and hence, has a minimal polynomial over K. Let $m(x) \in K[x]$ denote its minimal polynomial. Next, suppose that ℓ is also integral over R, so that ℓ also satisfies a monic polynomial with coefficients in R. Let $h(x) \in R[x]$ be a monic polynomial with $h(\ell) = 0$, and explain why $m(x)$ must divide $h(x)$ as elements in $K[x]$.

(b) By definition of L, every root of $m(x) \in K[x]$ lies in L. Continuing to assume that ℓ is integral over R, prove that every root $\lambda \in L$ of $m(x)$ is also integral over R.

(c) Conclude that the coefficients of $m(x) \in K[x]$ are integral over R.

Hint: As L contains all the roots of $m(x)$, we may write $m(x) = (x - \lambda_1) \cdots (x - \lambda_d)$ with each $\lambda_i \in L$. Describe how to obtain the coefficients of $m(x)$ in terms of these roots, and then use the previous part of this problem.

(d) Note that we have yet to invoke the assumption that R is normal. Use the normality of R here to conclude that $m(x)$ must lie in $R[x]$.