Worksheet 15: Going Down and Height

Throughout, R and S are commutative rings with unity, and k denotes a field.

Theorem (Going down)**.** *Suppose that $R \hookrightarrow S$ is an integral extension of domains, where R is normal. Then for every chain of prime ideals of R*

$$P_0 \subsetneq \cdots \subsetneq P_{n-1} \subsetneq P_n$$

and for every $Q_n \in \operatorname{Spec} S$ lying over P_n, there is a chain of primes in S ending with Q_n

$$Q_0 \subsetneq \cdots \subsetneq Q_{n-1} \subsetneq Q_n$$

such that for every $0 \leq i \leq n$, Q_i lies over P_i.

Definition. The *height* of a prime ideal $P \in \operatorname{Spec} R$ is the supremum of the lengths of all chains

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{n-1} \subsetneq P_n = P$$

of prime ideals of R contained in P.

Corollary. *Suppose that $R \hookrightarrow S$ is an integral extension of domains, where R is normal. Given any $Q \in \operatorname{Spec} S$, then if $P = Q \cap R$, the heights of P and Q coincide.*

1. **Warm-up.**
   (a) Prove that for any $P \in \operatorname{Spec} R$, its height equals the dimension of R_P.
   (b) Let $R \hookrightarrow S$ be an integral extension, and suppose that $Q \in \operatorname{Spec} S$ lies over $P \in \operatorname{Spec} R$.
      i. Prove that the height of P is at least the height of Q. *Hint*: Use Lying over.
      ii. Assuming the Going Down theorem, prove its above corollary.

The following two problems aim to better understand the hypotheses of the Going Down theorem.

2. Let $R = \mathsf{k}[x]$ and $S = \mathsf{k}[x, y]/\langle y^2 - y, xy\rangle$. Let Q be the principal ideal of S generated by $y - 1$.
   (a) Show that $R \hookrightarrow S$ is integral, and $x \in Q$.
   (b) Prove that $Q \in \operatorname{Spec} S$ and that S_Q is a field.
   (c) Show that $Q \cap R = \langle x \rangle$.
   (d) Compare the height of Q to the height of its contraction to R.
   (e) What does this not contradict the Going Down theorem?

3. Let $R = \mathsf{k}[s(1-s), t, st] \subseteq \mathsf{k}[s, t] = S$.
   (a) Show that this is an integral extension of domains. *Hint*: Consider $f(x) = x^2 - x + s(1-s)$.
   (b) Show that $Q = \langle 1 - s, t\rangle \in \operatorname{Spec} S$ contracts to $P = \langle s(1-s), t, st\rangle$, which is maximal.
   (c) Show that $\langle s \rangle \in \operatorname{Spec} S$ contracts to $P_0 = \langle s(1-s), st\rangle$.
   Hint: Which elements of R are not in P_0?
   (d) Show that no prime Q_0 contained in Q lies of P_0.
   (e) What does this not contradict the Going Down theorem?

4. **An easy preliminary result.** Suppose that $\varphi : R \to S$ is a ring homomorphism.

(a) Briefly explain why φ induces a map of rings $\Phi : R[x] \to S[x]$ that agrees with φ on $R \subseteq R[x]$. When $P \in \operatorname{Spec} R$ and $\varphi : R \to S = R/P$ is the quotient map, describe the induced map $R[x] \to (R/P)[x]$ in concrete terms.

(b) Prove that if R is a domain and $x^n = a(x)b(x)$ for some $a(x), b(x) \in R[x]$, then $a(x) = x^i$ and $b(x) = x^j$ for some $i, j \in \mathbb{N}$ that sum to n. Is this still true if R is not a domain?

(c) Conclude that if $g(x) \in R[x]$ is monic with all non-leading coefficients contained in some fixed prime ideal $P \in \operatorname{Spec} R$, and $g(x) = m(x)f(x)$ for some monic $m(x) \in R[x]$, then every non-leading coefficient of $m(x)$ must also lie in the same prime ideal P.
Hint: Apply the homomorphism $R[x] \to (R/P)[x]$ to the factorization $g = mf$.

Consider the following condition on an extension of rings $R \hookrightarrow S$:

For every $P \in \operatorname{Spec} R$ and $Q \in \operatorname{Spec} S$ lying over P, and for every $P_0 \in \operatorname{Spec} R$ strictly contained in P, there exists $Q_0 \in \operatorname{Spec} S$ strictly contained in Q that lies over P_0. (♠)

Lemma. *An integral extension $R \hookrightarrow S$ of domains, with R normal, satisfies* (**??**).

5. **Proving the lemma.** Let $P_0 \subsetneq P$ be prime ideals of R, and Q a prime ideal of S lying over P.
   (a) Show that $U = \{rs : r \in R \setminus P_0 \text{ and } s \in S \setminus Q\}$ is a multiplicatively closed subset of S.
   (b) Our next goal is to show that $U \cap (P_0 S) = \varnothing$. By way of contradiction, take $r \in R \setminus P_0$ and $s \in S \setminus Q$, and suppose that $rs \in P_0 S$.
      i. As S is integral over R, the element $rs \in S$ is integral over R. Explain why there exists a monic polynomial $h(x) \in R[x]$ with all non-leading coefficients contained in P_0 such that $h(rs) = 0$. *Hint*: Revisit the Lying over Worksheet.
      ii. Let L be the algebraic closure of K, the fraction field of R. Explain why our fixed element s lies in L. Conclude that $m(x) \in K[x]$, the minimal polynomial of $s \in L$, must actually lie in $R[x]$. *Hint*: Revisit the Normality Worksheet.
      iii. Set $g(x) = h(rx) \in R[x]$. Explain why $m(x)$ divides $g(x)$ in $K[x]$, and explain why this forces $m(x)$ to divide $g(x)$ in $R[x]$ as well.
      Hint: Because $m(x) \in R[x]$ is monic, you can divide $g(x)$ by $m(x)$ using the Division Algorithm to produce an expression $g(x) = m(x)q(x) + r(x)$ with $q, r \in R[x]$ and $\deg r(x) < \deg m(x)$, as you usually do over a field–convince yourself of this fact if it is unfamiliar to you. Then apply the uniqueness of quotients and remainders in the division algorithm for polynomials over a field.
      iv. Use Problem 4 to conclude that every non-leading coefficient of $m(x)$ lies in P_0.
      v. Obtain the contradiction by showing that your conclusion in (iv) is impossible.
      Hint: Evaluate $m(x)$ at $x = s$.
   (c) At this point, we have verified that U is a multiplicatively closed subset of S that does not intersect $P_0 S$. Explain why this implies that there exists a prime $Q_0 \in \operatorname{Spec} S$ containing $P_0 S$ such that $Q_0 \cap U = \varnothing$. Show that for such a prime, Q_0 lies over P_0 and is strictly contained in Q. Conclude that $R \hookrightarrow S$ satisfies property (**??**).
   Add hint; first worksheet!

6. Prove the Going Down theorem.

Definition. Given an extension of rings $R \hookrightarrow S$, we say that S is *torsion free* over R if whenever $r \in R$ and $s \in S$ are nonzero, then rs is a nonzero element of S.

Theorem (Going Down, Strong form). *Suppose that $R \hookrightarrow S$ is an integral extension, where R is normal and S is torsion free over R. Then for every chain of prime ideals of R, $P_0 \subsetneq \cdots \subsetneq P_{n-1} \subsetneq P_n$, and for every $Q_n \in \operatorname{Spec} S$ lying over P_n, there is a chain of primes in S with the form the form $Q_0 \subsetneq \cdots \subsetneq Q_{n-1} \subsetneq Q_n$ such that for every $0 \leq i \leq n$, Q_i lies over P_i.*

7. Construct an example of an integral extension $R \hookrightarrow S$ with R normal and S-torsion free over R, but not a domain. Thus, the strong version of Going Down is useful.

8. Consider $R = \mathsf{k}[x]$ and $S = \mathsf{k}[x, y]/\langle y^2 - y, xy\rangle$.
   (a) Show that the natural map $R \to S$ is an inclusion.
   Hint: Consider the ring map $\mathsf{k}[x, y] \to \mathsf{k}[x]$ given by $f(x, y) \mapsto f(x, 0)$. Explain why this induces a map $S \to R$ such that the composition $R \to S \to R$ is the identity on R.
   (b) Show that the extension $R \hookrightarrow S$ is integral.
   (c) Let Q be the ideal of S generated by the class of $y - 1$. Prove that Q is prime of height zero.
   Hint: To prove that Q is prime, consider S/Q. To prove that $\operatorname{ht}(Q) = 0$, argue by contradiction: Suppose $P \subsetneq Q$ is prime. Then P must contain zero, and so must contain the class of $y^2 - y = y(y - 1)$. Explain why this is impossible.
   (d) Show that Q contains the class of x. Conclude that Q lies over $\langle x\rangle$ in $\mathsf{k}[x]$, and prove that $\langle x\rangle$ has height one in $\mathsf{k}[x]$. *Hint*: Use that $\mathsf{k}[x]$ is PID to show that $\operatorname{ht}\langle x\rangle = 0$.
   (e) We have just shown that the Height Corollary to the Generalized Going Down Theorem fails in this instance. Why? Carefully explain what this says about which hypotheses in this theorem cannot be relaxed.