Worksheet 17: Dimension Theory for Finitely Generated k-algebras

Throughout, R and S are a commutative rings with unity, and k denotes a field.

Definition. A *saturated chain of prime ideals* of R of length d is a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d$ of primes of R such that for each i, there is no $Q \in \operatorname{Spec}(R)$ satisfying $P_i \subsetneq Q \subsetneq P_{i+1}$.

1. **Warm-up: Equivalent formulations of Krull dimension**. Recall that the Krull dimension of a ring R is the supremum over all n such that there exists an arbitrary chain in $\operatorname{Spec}(R)$ of length n.
   (a) (Krull dimension via saturated chains) Briefly explain why $\dim R$ equals the supremum of all n for which there exists a *saturated* chain of prime ideals of R of length n.
   (b) (Krull dimension of a domain) Briefly explain why if R is a domain, then $\dim R$ equals the supremum of all n for which there exists a saturated chain of primes starting at the zero ideal 0.
   (c) (Krull dimension and height) Formulate a description of the height of a prime ideal P of R in terms of saturated chains. In addition, describe $\dim R$ in terms of the heights of all maximal ideals of R. What does your description tell us about the Krull dimension of a local ring?

Theorem. *If R is a UFD, then every $P \in \operatorname{Spec}(R)$ with height one is a principal ideal.*

2. **Height one primes in a UFD.** Prove the above theorem.

   Hint: Why must P be nonzero? Also, recall that every irreducible element in a UFD is prime.

Theorem. *The Krull dimension of $\mathsf{k}[x_1, \dots, x_n]$ is n.*

3. **The dimension of a polymomial ring over a field.**
   (a) Recall that in a previous worksheet, you showed that $\dim \mathsf{k}[x_1, \dots, x_n] \geq n$ by explicitly constructing a chain of primes of length n, and notice that equality holds when $n = 0$.
   (b) We will prove the statement by induction on $n \geq 0$, and we checked the base case in (a). Consider our inductive hypothesis: Suppose $n > 0$, and assume the theorem is true for all polynomial rings over k in $n-1$ variables.
   (c) Consider a saturated chain of prime ideals $0 = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_m$ in $R = \mathsf{k}[x_1, \dots, x_n]$. Explain why it suffices to show that $m \leq n$ to complete the proof.
   (d) Using the fact that the chain we fixed is saturated, explain why P_1 must have height one. Conclude that P_1 must be principal.
   (e) Explain why, after changing variables, we may assume that P_1 is generated by a polynomial that is monic as a polynomial in x_n with coefficients in $\mathsf{k}[x_1, \dots, x_{n-1}]$.

   Hint: Apply a lemma from an earlier worksheet.
   (f) Explain why R/P_1 is integral over a polynomial ring over k in $n-1$ variables, and conclude that $\dim R/P_1$ equals $n-1$. What does this tell us about m? Conclude the proof.

Theorem. *If R is a finitely generated k-algebra that is also a domain, then the Krull dimension of R equals the transcendence degree of the field extension of k into $\operatorname{Frac}(R)$, the fraction field of R.*

Given a field extension $\mathsf{k} \subseteq \mathbb{L}$, recall that a *transcendence basis* for $\mathbb{L}$ over k is a subset Λ of $\mathbb{L}$ with the property that every finite subset of Λ is algebraically independent over k, which is maximal in the sense that no subset of $\mathbb{L}$ properly containing Λ satisfies this property. Transcendence bases always exist by Zorn's Lemma, and though it is a bit harder to show, all transcendence bases must all have the same cardinality, which is called the *transcendence degree* of the extension of $\mathsf{k} \subseteq \mathbb{L}$.

4. **Proof of theorem on Krull dimensions via field extensions.**
   (a) Prove that if $R \hookrightarrow S$ is an integral extension of domains, then the induced map on fraction fields $\mathrm{Frac}(R) \hookrightarrow \mathrm{Frac}(S)$ is an algebraic field extension.
   (b) Let $\mathsf{k} \hookrightarrow \mathbb{L}$ be an extension of fields. Prove that a subset Λ of $\mathbb{L}$ is a transcendence basis for this extension if and only if every finite subset of Λ is algebraically independent over k, and the extension $\mathsf{k}(\Lambda) \hookrightarrow \mathbb{L}$ is an algebraic field extension.
   (c) Prove the above theorem. *Hint*: Apply Noether Normalization.

Theorem. *If R is a finitely generated* k*-algebra that is also a domain, then every saturated chain of prime ideals of R starting at 0 and terminating at a maximal ideal have the same length, namely, the Krull dimension of R. In particular, the height of every maximal ideal of R equals* $\dim R$.

5. **Proof of theorem on heights of maximal ideals.** We proceed by induction on $\dim R$.
   (a) Verify the base case. *Hint*: The assumption that R is a domain is relevant.
   (b) Consider the following inductive hypothesis: Given $n > 0$, suppose that the theorem is true for all finitely-generated k-algebras that are domains and have dimension $n-1$. Let R be such a ring, but of dimension n, and consider a saturated chain of primes $0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_\ell$ with Q_ℓ a maximal ideal. To conclude our induction step, we must show that $\ell = n$.
      i. Explain why Q_1 must have height one.
      ii. Let $A = \mathsf{k}[x_1, \dots, x_d] \subseteq R$ be a Noether Normalization. Explain why the height of $P_1 := Q_1 \cap A$ is also one. *Hint*: Recall the Height Corollary from Going Down.
      iii. Explain why, after a change of variables, we can assume that P_1 is generated by a polynomial that is monic in x_n with coefficients in $\mathsf{k}[x_1, \dots, x_{n-1}]$. Conclude that A/P_1 is module finite over a polynomial ring in $n-1$ variables.
      iv. Explain why there is an induced inclusion $A/P_1 \hookrightarrow R/Q_1$ which is also module finite. Conclude that $\dim(R/Q_1) = \dim(A/P_1) = n-1$.
      v. Why does the inductive hypothesis apply to R/Q_1? What does this tells us about ℓ?
   (c) Complete the proof of the theorem.

6. Consider the ring $R = \mathsf{k}[x, y, z]/\langle xy, xz \rangle$.
   (a) Explain why every prime ideal of R either contains $\overline{x}$, or contains both $\overline{y}$ and $\overline{z}$.
   (b) Find two maximal ideals of R that have different heights.
   (c) Why does this not contradict the last statement of the previous theorem? What does this say about which of its hypotheses cannot be relaxed?

7. Let $A = \mathbb{Z}_P$, the localization of $\mathbb{Z}$ at the prime ideal $P = \langle 3 \rangle$, and set $R = A[t]$. You may use, without proof, the fact that R is a UFD.
   (a) Notice that A is *not* the ring obtained from $\mathbb{Z}$ by adding an inverse for 3. Describe A concretely a subset of $\mathbb{Q}$.
   (b) Verify that $\langle 3t-1 \rangle$ is a maximal ideal of R of height one.
   Hint: Doesn't $R/\langle 3t-1 \rangle = A[t]/\langle 3t-1 \rangle$ look like a localization of A?
   (c) Verify that $\langle 3, t \rangle$ is a maximal ideal of R of height at least two.
   (d) Why does this not contradict the last statement of the previous theorem? What does this say about which of its hypotheses cannot be relaxed?