Worksheet 18: Noetherian rings and modules

Throughout, R is a commutative ring with unity, and M is an R-module.

Definition. We call M *Noetherian* if every ascending chain of R-submodules $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of M is eventually stable, i.e., there exists an integer $N > 0$ for which $M_n = M_{n+1}$ for all $n \geq N$. We call R is a *Noetherian ring* (or just call R *Noetherian*) if it is Noetherian as a module over itself.

1. **Warm-up: Noetherian rings.**
   (a) Explain what it means for R to be Noetherian, in terms of ascending chains.
   (b) Verify that fields and PIDs are Noetherian rings.
   (c) Must every UFD be Noetherian? *Hint*: Observe that the ascending union of UFDs is a UFD.
   (d) Explain why, if R is Noetherian, then so is R/I, for every ideal $I \subseteq R$.
   (e) Explain why, if R is Noetherian, then so is $\mathcal{W}^{-1}R$, for every multiplicative set $\mathcal{W} \subseteq R$.

2. **First examples and counterexamples.**
   (a) Find a pair of nonzero R and M such that: (i) R and M are both Noetherian, (ii) R is Noetherian, but M is not, (iii) M is not Noetherian, but R is not, and (iv) neither R nor M is Noetherian.
   (b) Let R be the $\mathbb{R}$-algebra consisting of all continuous functions $f : [0, 1] \to \mathbb{R}$, with addition and multiplication of functions defined pointwise. Prove that R is not a Noetherian ring.

   Hint: Fix a proper closed subinterval I of the unit interval, and consider the set of all $f \in R$ vanishing at each point of I.

3. **Characterizations of Noetherian rings.** Prove that the following conditions are equivalent.
   (a) R is Noetherian.
   (b) Every non-empty family of ideals $\{I_i\}_{i \in \Sigma}$ of R has a maximal element, i.e., there exists $j \in \Sigma$ such that whenever $I_j \subseteq I_i$ for some $i \in \Sigma$, then $I_j = I_i$. *Note that this condition does not require that any member of the family be contained in such a maximal element, besides the element itself.*
   (c) Every ascending chain of *finitely generated* ideals of R is eventually stable.
   (d) For every ideal $I \subseteq R$, and for every subset $\mathcal{A} \subseteq R$ with $I = \langle \mathcal{A} \rangle$, there exists a *finite* subset $\mathcal{B} \subseteq \mathcal{A}$ such that $I = \langle \mathcal{B} \rangle$.
   (e) Every ideal of R is finitely generated.

(∗) **Characterizations of Noetherian modules.** Convince yourself that the following conditions on an R-module M are equivalent; you are not required to write down the proof.
   (a) M is a Noetherian R-module.
   (b) Every non-empty family of submodules $\{M_i\}_{i \in \Sigma}$ of M has a maximal element, i.e., there exists $j \in \Sigma$ such that if $i \in \Sigma$ and $M_j \subseteq M_i$, then $M_j = M_i$.
   (c) Every ascending chain of *finitely generated* submodules of M is eventually stable.
   (d) For every submodule $N \subseteq M$, and for every subset $\mathcal{A} \subseteq M$ that generates N, there exists a *finite* subset $\mathcal{B} \subseteq \mathcal{A}$ that also generates N.
   (e) Every submodule of M, including M itself, is finitely generated.

Proposition. *If $0 \to N \to M \to Q \to 0$ is a short exact sequence of R-modules, then M is a Noetherian R-module if and only if both N and Q are Noetherian R-modules.*

Corollary. *Suppose that R is Noetherian. Then an R-module M is Noetherian if and only if M is finitely generated. In particular, every submodule of a finitely generated module of a Noetherian ring is also finitely generated.*

4. **Exact sequences and Noetherian modules.**
   (a) After fixing names for the homomorphism appearing in the above proposition as $\varphi : N \to M$ and $\psi : M \to Q$, prove the following **Lemma**: If $M_0 \subseteq M_1 \subseteq M$ are submodules such that $M_0 \cap \varphi(N) = M_1 \cap \varphi(N)$ and $\psi(M_0) = \psi(M_1)$, then $M_0 = M_1$.
   (b) Prove the proposition. *Hint*: Apply your lemma for the " $\Longleftarrow$ " implication.
   (c) Prove the corollary, which is powerful in that it provides a concrete, simplified characterization for Noetherianity of an R-module in the special case that R is a Noetherian ring.
   Hint: For the " $\Longleftarrow$ " implication, first prove that the finitely generated free module R^n is Noetherian for each $n \geq 1$. If M is a finitely generated R-module, can you fit it into a short exact sequence that involves R^n, for some n?
   (d) Show that the last statement in the corollary can fail by when R is not Noetherian.

Theorem (Cohen). *R is Noetherian if and only if every **prime** ideal of R is finitely generated.*

5. **Noetherian rings and prime ideals.**
   (a) Prove the above theorem by applying a result from the first worksheet.
   (b) It is natural to ask where else we might be able to replace *ideal* with *prime ideal* to obtain a relaxed characterization of Noetherian rings. If every ascending chain of prime ideals of R is eventually stable, then must R be Noetherian?
   Hint: Can you find a non-Noetherian ring with very few prime ideals?

Theorem (Hilbert Basis Theorem). *If R is a Noetherian ring, then so is $R[x]$.*

Definition. A ring S has *finite type over R* if S is a finitely generated R-algebra, and S is said to be *essentially of finite type over R* if it is the localization of a finitely generated R-algebra at some multiplicatively closed set.

Corollary. *Every ring essentially of finite type over a Noetherian ring is also Noetherian. In particular, every ring that is essentially of finite type over a field is Noetherian.*

6. **Proving the Hilbert Basis Theorem.**
   (a) Suppose that R is arbitrary. Given a nonzero polynomial $f(x) = a_0 + a_1 x + \cdots a_n x^n$ with each $a_i \in R$, and $a_n \neq 0$, let $\mathrm{LT}(f) = a_n$ denote the leading term of f. We adopt the convention $\mathrm{LT}(0) = 0$. Given an ideal I of $R[x]$, prove that $\mathrm{LT}(I) := \{\mathrm{LT}(f) : f \in I\}$ is a (possibly improper) ideal of R.

   Now assume that R is Noetherian, and that I is an ideal of $R[x]$.
   (b) Explain why there exist $f_1, \ldots, f_\ell \in I$ such that $\mathrm{LT}(I) = \langle \mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_\ell) \rangle$.
   (c) Let $m = \max\{\deg(f_1), \ldots, \deg(f_\ell)\}$. Show that $N = \{g \in I : \deg(g) \leq m\} \cup \{0\}$ is an R-submodule of $R[x]$, but not an ideal of $R[x]$.
   (d) Prove that N is a finitely generated R-module.
   Hint: As R is assumed to be Noetherian, we know from an earlier corollary that *if* we can find a finitely generated (and hence, Noetherian) R-module M containing N, then this will force N to be finitely generated. Seek a natural choice of M that satisfies this condition.

(e) Prove that every element $f \in I$ can be written as $h + g$ with $h \in \langle f_1, \ldots, f_\ell \rangle$ and $g \in N$.

Hint: Induce on $\deg(f)$; the case when $\deg(f) \leq m$ is trivial.

(f) Conclude the proof of the Hilbert Basis Theorem by explaining why I is finitely generated.

(g) Observe that almost every explicitly defined ring we've seen in this course (the exceptions being rings such as the $\mathbb{R}$-algebra of continuous function $f : [0,1] \to \mathbb{R}$, and polynomial rings with countably many variables) are essentially of finite type over a field, or over a PID. Then prove the above corollary, showing that each of these rings is Noetherian.

Hint: Deduce a generalization of the Hilbert Basis Theorem that involves finitely many variables, and apply some results from the Warm-up.