Worksheet 19: Nakayama's Lemma

Throughout, R is a commutative ring with unity, k is a field, and M is an R-module. Recall the definition of a local ring from the *Localization of Rings* worksheet.

Definition. A *local ring* is a ring with a unique maximal ideal. If R is local with maximal ideal $\mathfrak{m}$, we often denote it concisely by the pair $(R, \mathfrak{m})$ if we wish to refer to its maximal ideal.

Definition. If R is local with maximal ideal $\mathfrak{m}$, then the *residue field* of R is $\mathsf{k} = R/\mathfrak{m}$. If we wish to refer to its residue field, we often denote the local ring concisely as the triple $(R, \mathfrak{m}, \mathsf{k})$.

1. **Warm-up: Local rings.**
   (a) If $(R, \mathfrak{m})$ is a local ring, what are its units?
   (b) If I is a proper ideal of R and every element of $R \setminus I$ is a unit, prove that I is a maximal ideal and (R, I) is local. Then use this, with (a), to characterize local rings in terms of their units.
   (c) Recall from #4 in the *Localization of Rings* worksheet that if P is a prime ideal of R, then the localization R_P is a local ring. Compute the residue fields of $\mathsf{k}[x, y, z]_{\langle x,y,z\rangle}$ and $\mathbb{Z}_{\langle 2\rangle}$. You might want to look back on #5 from the *Localization of Rings* worksheet.
   (d) Let $\mathsf{k}[\![x]\!]$ denote the set of formal expressions fo the form $\sum\limits_{n=0}^{\infty} a_n x^n$, where all $a_i \in \mathsf{k}$. Under the natural addition and multiplication of these power series expressions, convince yourself that $\mathsf{k}[\![x]\!]$ is a ring, which we call the *ring of formal power series* in x over k. Then prove that it is a local ring using (b) by first conjecturing its unique maximal ideal $\mathfrak{m}$, and then constructing an inverse for any $f \notin \mathfrak{m}$ inductively in terms of its coefficients.
   (e) Convince yourself that for any $n \geq 0$, the ring $\mathsf{k}[\![x_1, \ldots, x_n]\!]$ of formal power series in $x_1, \ldots, x_n$ over k is a local ring. What is its unique maximal ideal $\mathfrak{m}$, and its residue field? To show that $f \notin \mathfrak{m}$ is indeed a unit, first write $f = f_0 + f_1 + \cdots$, where $f_d \in \mathsf{k}[x_1, \ldots, x_n]$ is homogeneous of degree d, and proceed similarly as you did in (d) to ensure the existence of $f^{-1} \in \mathsf{k}[\![x_1, \ldots, x_n]\!]$.

Definition. If I is an ideal of R, and M is an R-module, then IM denotes the R-submodule of M consisting of finite sums of elements of the form au, where $a \in I$ and $u \in M$.

Theorem (Nakayama's Lemma, Version 1)**.** *Let $(R, \mathfrak{m})$ be a local ring. If M is a finitely-generated R-module and $\mathfrak{m}M = M$, then $M = 0$.*

2. **Proving Nakayama's Lemma, V1**. Fix $(R, \mathfrak{m})$ local.
   (a) Prove Nakayama's Lemma, V1 in the case that M is generated by one element.
   Hint: If $M = R\langle u\rangle$, explain where there exists $a \in \mathfrak{m}$ such that $au = u$. Simplify, and appeal to the Warm-up to see that $a - 1$ is a unit.
   (b) Suppose $n > 1$ and $u_1, \ldots, u_n \in M$ generate M. Let N be the submodule of M generated by u_n. If $\mathfrak{m}M = M$, show that $L = M/N$ satisfies $\mathfrak{m}L = L$.
   (c) Finish the proof of Nakayama's Lemma, V1 by inducing on the number of generators of M.

3. **The quotient $M/\mathfrak{m}M$.** Suppose that $(R, \mathfrak{m}, \mathsf{k})$ is local, and M is an R-module.
   (a) Construct an R-module isomorphism $\mathsf{k} \otimes_R M \cong M/\mathfrak{m}M$. follows from tensor product worksheet

(b) Describe the natural k-vector space structure on each of $\mathsf{k} \otimes_R M$ and $M/\mathfrak{m}M$, and verify that your map in (a) is also a k-vector space isomorphism.

(c) If M is a finitely generated R-module, prove that $M/\mathfrak{m}M$ is finite dimensional k-vector space.

(d) Compute the k-vector space dimension of $M/\mathfrak{m}M$ in the following cases.

i. $(R, \mathfrak{m}, \mathsf{k})$ is an arbitrary local ring, and $M = R^{\oplus 5}$.
Hint: Tensor products commute with direct sums.

ii. $R = \mathsf{k}[x, y, z]_{\langle x,y,z\rangle}$ and $M = \mathfrak{m} = \langle x, y, z\rangle$.

iii. $R = \mathsf{k}[\![x, y, z]\!]/\langle x^2 + y^3 + z^5\rangle$ and $M = \mathfrak{m} = \langle \overline{x}, \overline{y}, \overline{z}\rangle$. (Why is $(R, \mathfrak{m})$ local here?)

Theorem (Nakayama's Lemma, Version 2)**.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring. Let M be a finitely generated R-module, and fix $u_1, \dots, u_n \in M$. Then $u_1, \cdots, u_n$ generate M as an R-module if and only if the images $\overline{u}_1, \dots, \overline{u}_n \in M/\mathfrak{m}M$ span $M/\mathfrak{m}M$ as a k-vector space.*

4. Prove Nakayama's Lemma, V2.

   Hint: You proved the " $\Longrightarrow$ " implication in #2. For the " $\Longleftarrow$ " implication, let N be the R-module generated by $u_1, \dots, u_d$. Show that $M = N + \mathfrak{m}M$, and then explain why this allows us to apply Nakayama's Lemma to the quotient $L = M/N$.

5. **Consequences of Nakayama's Lemma**.

(a) Suppose that for some proper ideal I of a local ring R, $IM = M$ for some finitely generated R-module M. Prove that $M = 0$.

(b) Given an R-module M, we call a subset $\mathcal{A} \subseteq M$ a *minimal generating set* if $\mathcal{A}$ generates M, but no proper subset of $\mathcal{A}$ generates M. Suppose that $(R, \mathfrak{m}, \mathsf{k})$ is local, and M is a finitely generated R-module. Prove that $\mathcal{A}$ is a minimal generating set for M if and only if the image of $\mathcal{A}$ in the quotient $M/\mathfrak{m}M$ forms a basis for the k-vector space $M/\mathfrak{m}M$. Conclude that any two minimal generating sets of a finitely generated module over a local ring have the same cardinality.

(c) Suppose that if D is a (not necessarily local) domain that is not a field. Prove that its fraction field $\mathrm{Frac}(D)$ is *not* finitely generated as a D-module.

Hint: If R is the localization of D at any prime ideal of Dexplain why $D \subseteq R \subseteq \mathrm{Frac}(R)$. If $\mathrm{Frac}(R)$ is finitely generated as an D-module, explain why $\mathrm{Frac}(R)$ must be finitely generated as an R-module. Finally, apply Nakayama's Lemma to the finitely generated R-module $\mathrm{Frac}(R)$ to get a contradiction. Where did you use that D was not a field?

(d) Let $\varphi : M \to N$ be a homomorphism of finitely generated R-modules, where $(R, \mathfrak{m})$ is local.

i. Prove that φ is surjective if and only if the induced map $\overline{\varphi} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ of k-vector spaces is surjective.

ii. The corresponding statement obtained by replacing *surjective* with *injective* is false. In fact, both implications are false! Construct examples to demonstrate this.
Hint: Let $R = \mathbb{Z}_{\langle p\rangle}$ be the localization of $\mathbb{Z}$ at the ideal generated by a nonzero prime integer p, and consider the maps $R \xrightarrow{p} R$ and $R \to R/\langle p\rangle$.

6. **The hypotheses of Nakayama's Lemma.**

(a) Let $R = \mathsf{k}[\![x]\!]$, and let M be the R-algebra $M = R_x = \mathsf{k}[\![x]\!][x^{-1}]$, i.e., the localization of R at the multiplicative set $\{1, x, x^2, \dots\}$. Explain why the maximal ideal $\mathfrak{m} = \langle x\rangle \subseteq R$ satisfies $\mathfrak{m}M = M$. What does this not contradict Nakayama's Lemma, V1?

(b) Demonstrate that the hypothesis that R is local in Nakayama's Lemma, V1 cannot be removed by constructing a non-local ring R and a nonzero finitely generated R-module M such that $\mathfrak{m}M = M$ for some maximal ideal $\mathfrak{m}$ of R. *Hint*: There are simple examples, e.g., try $R = \mathbb{R}[x]$.