Worksheet 20: The I-adic Topology and Completion

Throughout, R is a commutative ring with unity, I is a proper ideal of R, M is an R-module, and k denotes a field. Let $R^{\mathbb{N}}$ denote the ring of all sequences $(r_n)_{n\in\mathbb{N}}$ with terms in R.

Definition. Consider a sequence $(r_n)_{n\in\mathbb{N}}$ in $R^{\mathbb{N}}$.

1. The sequence is *Cauchy in the I-adic topology* if for every $t \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that $r_n - r_m \in I^t$ whenever $n, m \geq d$. We will denote the set of all Cauchy sequences by $\mathfrak{C}_I(R)$.
2. The sequence *converges to zero* if for every $t \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that whenever $n \geq d$, we have that $r_n \in I^t$. We denote the set of sequences that converge to zero by $\mathfrak{C}^0_I(R)$.

1. **Cauchy sequences.**
   (a) Show that the subset $\mathfrak{C}_I(R)$ is a subring of $R^{\mathbb{N}}$.
   (b) Show that the map $R \to \mathfrak{C}_I(R)$ sending r to the constant sequence (r) is a ring homomorphism, making $\mathfrak{C}_I(R)$ into an R-algebra.
   (c) Prove that $\mathfrak{C}^0_I(R)$ is an ideal of $\mathfrak{C}_I(R)$.
   (d) Observe that a subsequence of a Cauchy sequence is Cauchy, and differs from the original one by a sequence that converges to zero.

Definition. The *completion of R in the I-adic topology* is the ring $\widehat{R}^I = \mathfrak{C}_I(R)/\mathfrak{C}^0_I(R)$.

2. Each element of $\widehat{R}^I$ is an equivalence class of Cauchy sequences that differ by ones converging to zero, which we call a *limit of a Cauchy sequence in R*. Compare this to the construction of the real numbers as the collection of limits of Cauchy sequences of rational numbers.

3. Define a natural ring map $R \to \widehat{R}^I$. What is its kernel? If $(R, \mathfrak{m})$ is Noetherian and local, can you show that $R \to \widehat{R}^{\mathfrak{m}}$ is injective?

4. Let $R = \mathbb{Z}$ and $I = \langle p \rangle$ for p a prime integer. Show that

$$\left(\sum_{i=0}^{n} p^i\right)_{n\in\mathbb{N}} = (1, 1+p, 1+p+p^2, 1+p+p^2+p^3, \ldots)$$

   is a Cauchy sequence for the $\langle p \rangle$-adic (also called the p-adic) topology on $\mathbb{Z}$. Show also that it represents the inverse of $1-p$ in the completion $\widehat{\mathbb{Z}}^{\langle p \rangle}$[1].

5. Let $R = \mathsf{k}[x, y]$ and $I = \langle x, y \rangle$.
   (a) Show that

$$\left(\sum_{i=0}^{n} r_i x^i y^{i+2}\right)_{n\in\mathbb{N}} = (r_0 y^2, r_0 y^2 + r_1 x y^3, r_0 y^2 + r_1 x y^3 + r_2 x^2 y^4, \ldots)$$

   is a Cauchy sequence in the I-adic topology
   (b) Find another Cauchy sequence with the same limit as the one in (a) (i.e., it should represent the same element of $\widehat{R}^I$, or equivalently, differ by a sequence converging to zero).

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[1]This is the ring of *p-adic integers*, used often in number theory, and often denoted $\widehat{\mathbb{Z}}_p$, $\mathbb{Z}_p$, or $\mathbf{Z}_p$.

(c) Describe a natural ring map $\mathsf{k}[\![x,y]\!] \to \widehat{R}^I$ taking a power series $\sum_{i=0}^{\infty} r_{ij}x^i y^j$ to an equivalence class of Cauchy sequences. Can you show that it's injective?
Hint: To come up with the map, think *successive truncations*.

6. Explain how to think of an element of $\widehat{R}^I$ (non-uniquely) as an infinite sum $\sum_{n=0}^{\infty} x_n$, where $x_n \in I^n$. Try using this idea to prove that your map from 5(c) is surjective.

Theorem. *Let $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ be the homogeneous maximal ideal of $S = \mathsf{k}[x_1, \dots, x_n]$. Then $\widehat{S}^{\mathfrak{m}} \cong \mathsf{k}[\![x_1, \dots, x_n]\!]$. Furthermore, given $g_1, \dots, g_\ell \in S = \mathsf{k}[x_1, \dots, x_n] \subseteq \mathsf{k}[\![x_1, \dots, x_n]\!] = \widehat{S}^{\mathfrak{m}}$,*

$$\textit{if } R = \mathsf{k}[x_1, \dots, x_n]/\langle g_1, \dots, g_t \rangle, \textit{ then } \widehat{R}^I \cong \mathsf{k}[\![x_1, \dots, x_n]\!]/\langle g_1, \dots, g_t \rangle.$$

7. Assume the theorem. Let $R = \mathbb{R}[x,y]/\langle y^2 - x^2 - x^3 \rangle$, and let $\mathfrak{m} = \langle x, y \rangle \subseteq R$.
   (a) Prove that R is a domain.
   (b) Show that $\widehat{R}^{\mathfrak{m}}$ is not a domain. *Hint.* Find a power series representing $\sqrt{1+x}$.
   (c) Sketch the curve $y^2 = x^3 + x^2$. There is a sense that the minimal primes of $\widehat{R}^{\mathfrak{m}}$ can be identified by focusing in a very small neighborhood of the origin. How is this reflected in your picture?

8. Let $(r_n)_{n\in\mathbb{N}} \in \mathfrak{C}_I(R)$ be a Cauchy sequence, and fix $t \in \mathbb{N}$.
   (a) Explain why the sequence of residues $(r_n \bmod I^t)_{n\in\mathbb{N}}$ is eventually constant as $n \to \infty$.
   (b) Use (a) to show that there is a surjective ring homomorphism $\mathfrak{C}_I(R) \twoheadrightarrow R/I^t$.
   (c) Use (b) to show that there is a surjective ring homomorphism $\widehat{R}^I \twoheadrightarrow R/I^t$.
   (d) Show that for any $s \leq t$, the map in (c) for s is the composition $\widehat{R}^I \twoheadrightarrow R/I^t \twoheadrightarrow R/I^s$.

Definition. Consider a sequence $(u_n)_{n\in\mathbb{N}}$ whose terms come from a fixed R-module M. We call (u_n) *Cauchy in the I-adic topology* if for every $t \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that whenever $n, m \geq d$, we have that $u_n - u_m \in I^t M$. We will denote the set of all Cauchy sequences with terms in M by $\mathfrak{C}_I(M)$. The sequence (u_n) *converges to zero* if for every $t \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that $u_n \in I^t M$ whenever $n \geq d$. We denote the set of all sequences with terms in M that converge to zero by $\mathfrak{C}_I^0(M)$.

9. **Cauchy sequences in M.**
   (a) Prove that $\mathfrak{C}_I(M)$ is a module over the ring $\mathfrak{C}_I(R)$.
   (b) Prove that $\mathfrak{C}_I^0(M)$ forms a $\mathfrak{C}_I(R)$-submodule of $\mathfrak{C}_I(M)$.
   (c) Prove that the abelian group $\mathfrak{C}_I(M)/\mathfrak{C}_I^0(M)$ has the structure of an $\widehat{R}^I$-module.

Definition. We call the $\widehat{R}^I$-module $\widehat{M}^I = \mathfrak{C}_I(M)/\mathfrak{C}_I^0(M)$ the *completion of M along I*.